

CANONICAL MEASURES AND KÄHLER-RICCI FLOW ¹

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Abstract

We show that the Kähler-Ricci flow on an algebraic manifold of positive Kodaira dimension and semi-ample canonical line bundle converges to a unique canonical metric on its canonical model. It is also shown that there exists a canonical measure of analytic Zariski decomposition on an algebraic manifold of positive Kodaira dimension. Such a canonical measure is unique and invariant under birational transformations under the assumption of the finite generation of canonical rings.

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1 Introduction

It has been the subject of intensive study over the last few decades to study the existence of Kähler-Einstein metrics on a compact Kähler manifold, following Yau's solution to the Calabi conjecture (cf. [Ya2], [Au], [Ti2], [Ti3]). The Ricci flow (cf. [Ha, Ch]) provides a canonical deformation of Kähler metrics in Kähler geometry. Cao [Ca] gave an alternative proof of the existence of Kähler-Einstein metrics on a compact Kähler manifold with trivial or negative first Chern class by the Kähler-Ricci flow. However, most algebraic manifolds do not have a definite or trivial first Chern class. It is a natural question to ask if there exist any well-defined canonical metrics on these manifolds or on varieties canonically associated to them. Tsuji [Ts1] applied the Kähler-Ricci flow and proved the existence of a canonical singular Kähler-Einstein metric on a minimal algebraic manifold of general type. It was the first attempt to relate the Kähler-Ricci flow and canonical metrics to the minimal model program. Since then, many interesting results have been achieved in this direction. The long time existence of the Kähler-Ricci flow on a minimal algebraic manifold with any initial Kähler metric is established in [TiZha]. The regularity problem of the canonical singular Kähler-Einstein metrics on minimal algebraic manifolds of general type is intensively studied in [Zh, EyGuZe1].

In this paper, we propose a program of finding canonical measures on algebraic varieties of positive Kodaira dimension. Such a canonical measure can be considered as a birational invariant and it induces a canonical singular metric on the canonical model, generalizing the notion of Kähler-Einstein metrics.

Let X be an n -dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form ω on X . In local coordinates z_1, \dots, z_n , we can write ω as

$$\omega = \sqrt{-1} \sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

where $\{g_{i\bar{j}}\}$ is a positive definite hermitian matrix function. Consider the normalized Kähler-Ricci flow

$$\begin{cases} \frac{\partial \omega(t, \cdot)}{\partial t} = -\text{Ric}(\omega(t, \cdot)) - \omega(t, \cdot), \\ \omega(0, \cdot) = \omega_0, \end{cases} \quad (1.1)$$

where $\omega(t, \cdot)$ is a family of Kähler metrics on X , $\text{Ric}(\omega(t, \cdot))$ denotes the Ricci curvature of $\omega(t, \cdot)$ and ω_0 is a given Kähler metric.

Let X be a minimal algebraic manifold. If the canonical line bundle K_X of X is ample and ω_0 represents $[K_X]$, it is proved in [Ca] that (1.1) has a global solution $\omega(t, \cdot)$ for all $t \geq 0$ and $\omega(t, \cdot)$ converges to a unique Kähler-Einstein metric on X . Tsuji showed in [Ts1] that (1.1) has a global solution $\omega(t, \cdot)$ under the assumption that the initial Kähler class $[\omega_0] > [K_X]$. This additional assumption was removed in [TiZha], moreover, if K_X is also big, $\omega(t, \cdot)$ converges to a singular Kähler-Einstein metric with locally bounded Kähler potential as t tends to ∞ (see [Ts1, TiZha]).

If K_X is not big, the Kodaira dimension of X is smaller than its complex dimension. In particular, when X is a minimal Kähler surface of Kodaira dimension 1, it must be a minimal elliptic surface and does not admit any Kähler-Einstein current in $-c_1(X)$, with bounded local

potential smooth outside a subvariety. Hence, one does not expect that $\omega(t, \cdot)$ converges to a smooth Kähler-Einstein metric outside a subvariety of X in general.

Let $f : X \rightarrow X_{can}$ be a minimal elliptic surface of $\text{kod}(X) = 1$. Suppose all the singular fibres are given by $X_{s_1} = f^{-1}(s_1), \dots, X_{s_k} = f^{-1}(s_k)$ of multiplicity $m_i \in \mathbb{N}, i = 1, \dots, k$. In [SoTi], the authors proved that the Kähler-Ricci flow on X converges for any initial Kähler metric to a positive current ω_{can} on the canonical model X_{can} satisfying

$$\text{Ric}(\omega_{can}) = -\omega_{can} + \omega_{WP} + 2\pi \sum_{i=1}^k \frac{m_i - 1}{m_i} [s_i], \quad (1.2)$$

where ω_{WP} is the induced Weil-Petersson metric and $[s_i]$ is the current of integration associated to the divisor s_i on X_{can} . ω_{can} is called a generalized Kähler-Einstein metric on X_{can} . Moreover, the Kähler-Ricci flow is collapsing onto X_{can} exponentially fast with uniformly bounded scalar curvature away from the singular fibres.

The first result of this paper is to generalize the above convergence result on the Kähler-Ricci flow to algebraic manifolds of positive Kodaira dimension and semi-ample canonical bundle.

Let X be an n -dimensional algebraic manifold of Kodaira dimension $0 < \kappa < n$. We assume that canonical line bundle K_X is semi-ample, then the canonical ring $R(X, K_X)$ is finitely generated and the pluricanonical system induces an algebraic fibre space $f : X \rightarrow X_{can}$. Each nonsingular fibre of f is a nonsingular Calabi-Yau manifold. We denote by X_{can}° the set of all nonsingular points $s \in X_{can}$ such that $f^{-1}(s)$ is a nonsingular fibre and let $X^\circ = f^{-1}(X_{can}^\circ)$. The L^2 -metric on the moduli space of nonsingular Calabi-Yau manifolds induces a semi-positive $(1, 1)$ -form ω_{WP} of Weil-Petersson type on X_{can}° . We will study the Kähler-Ricci flow starting from any Kähler metric and describe its limiting behavior as time goes to infinity.

Theorem A *Let X be a nonsingular algebraic variety with semi-ample canonical line bundle K_X and so X admits an algebraic fibration $f : X \rightarrow X_{can}$ over its canonical model X_{can} . Suppose $0 < \dim X_{can} = \kappa < \dim X = n$. Then for any initial Kähler metric, the Kähler-Ricci flow (1.1) has a global solution $\omega(t, \cdot)$ for all time $t \in [0, \infty)$ satisfying:*

1. $\omega(t, \cdot)$ converges to $f^*\omega_{can} \in -2\pi c_1(X)$ as currents for a positive closed $(1, 1)$ -current ω_{can} on X_{can} with continuous local potential.
2. ω_{can} is smooth on X_{can}° and satisfies the generalized Kähler-Einstein equation on X_{can}°

$$\text{Ric}(\omega_{can}) = -\omega_{can} + \omega_{WP}. \quad (1.3)$$

3. for any compact subset $K \in X_{can}^\circ$, there is a constant C_K such that

$$\|R(t, \cdot)\|_{L^\infty(f^{-1}(K))} + e^{(n-\kappa)t} \sup_{s \in K} \|\omega^{n-\kappa}(t, \cdot)\|_{L^\infty(X_s)} \leq C_K, \quad (1.4)$$

where $X_s = f^{-1}(s)$.

Therefore, the Kähler-Ricci collapses onto the canonical model with bounded scalar curvature away from the singular fibres and the volume of each nonsingular fibre tends to 0 exponentially fast. In fact, the local potential of $\omega(t, \cdot)$ converges on X° locally in C^0 -topology (cf.

Proposition 5.4). It should also converge locally in $C^{1,1}$ -topology on X° as in the surface case (cf. [SoTi]) and this will be studied in detail in a forthcoming paper.

Similar phenomena also appears in the real setting as a special type-III Ricci flow solution without the presence of singular fibres. It is discovered and intensively studied in [Lo].

The abundance conjecture in algebraic geometry predicts that the canonical line bundle is semi-ample if it is nef. If the abundance conjecture is true, then Theorem A immediately implies that on all nonsingular minimal models of positive Kodaira dimension the Kähler-Ricci flow converges to a unique canonical metric on their canonical model.

In general, the canonical line bundle of an algebraic manifold of positive Kodaira dimension is not necessarily semi-ample or even nef. The minimal model program in birational geometry deals with the classification of algebraic varieties and aims to choose a minimal model in each birational equivalence class. Tsuji claimed in [Ts3] that there exists a singular Kähler-Einstein metric of analytic Zariski decomposition on algebraic manifolds of general type without assuming the finite generation of the canonical ring. Such a metric is constructed through a family of Kähler-Einstein metrics as the limits of a parabolic Monge-Ampère equation of Dirichlet type. The approach is interesting but rather complicated. The recent exciting development in the study of degenerate complex Monge-Ampère equations (cf. [Kol1, Zh, EyGuZe1]) enables the authors to give an independent and correct proof.

Theorem B.1 *Let X be an algebraic manifold of general type. Then there exists a measure Ω_{KE} on X such that*

1. (K_X, Ω_{KE}^{-1}) is an analytic Zariski decomposition.
2. Let $\omega_{KE} = \sqrt{-1}\partial\bar{\partial}\log\Omega_{KE}$ be the closed positive $(1,1)$ current on X . Then there exists a non-empty Zariski open subset U of X such that $\text{Ric}(\omega_{KE}) = \sqrt{-1}\partial\bar{\partial}\log(\omega_{KE})^n$ is well-defined on U and

$$\text{Ric}(\omega_{KE}) = -\omega_{KE}.$$

The proof of Theorem B.1 is given in Section 4.3. The existence of such a canonical Kähler-Einstein metric is also considered by Siu in [Si1] as an alternative approach to attack the problem of the finite generation of canonical rings. A degenerate Monge-Ampère equation of Dirichlet type is considered and the solution is expected to be unique. Indeed, if the canonical rings are finitely generated, such a solution coincide with the Kähler-Einstein metrics constructed in Theorem B.1. We hope that Theorem B.1 might help to gain more understanding of the finite generation of canonical rings from an analytic point of view. Theorem B.1 can be generalized to algebraic manifolds of positive Kodaira dimension.

Theorem B.2 *Let X be an n -dimensional algebraic manifold of Kodaira dimension $0 < \kappa < n$. There exists a measure Ω_{can} on X bounded above such that (K_X, Ω_{can}^{-1}) is an analytic Zariski decomposition. Let $\Phi^\dagger : X^\dagger \rightarrow Y^\dagger$ be any Iitaka fibration of X with $\pi^\dagger : X^\dagger \rightarrow X$ and $\Omega^\dagger = (\pi^\dagger)^*\Omega_{can}$. Then*

1. $(K_{X^\dagger}, (\Omega^\dagger)^{-1})$ is an analytic Zariski decomposition.

2. *There exists a closed positive $(1, 1)$ -current ω^\dagger on Y^\dagger such that $(\Phi^\dagger)^* \omega^\dagger = \sqrt{-1} \partial \bar{\partial} \log \Omega^\dagger$ on a Zariski open set of X^\dagger . Furthermore, we have*

$$(\omega^\dagger)^\kappa = (\Phi^\dagger)_* \Omega^\dagger, \quad (1.5)$$

on Y^\dagger and so on a Zariski open set of Y^\dagger ,

$$\text{Ric}(\omega^\dagger) = -\omega^\dagger + \omega_{WP}. \quad (1.6)$$

The definition of ω_{WP} in Theorem B.2 is given in Section 4.4 and it is a generalization of the Weil-Petersson form induced from an algebraic deformation space of algebraic manifolds of Kodaira dimension 0.

In fact, the hermitian metric Ω_{can}^{-1} (also Ω_{KE}^{-1}) on K_X constructed in the proof of Theorem B.2 (also Theorem B.1) has stronger properties than the analytic Zariski decomposition. Let

$$\Psi_{X,\epsilon} = \sum_{m=1}^{\infty} \sum_{j=0}^{d_m} \epsilon_{m,j} |\sigma_{m,j}|^{\frac{2}{m}},$$

where $\{\sigma_{m,j}\}_{j=0}^{d_m}$ spans $H^0(X, mK_X)$ and $\{\epsilon_{m,j} > 0\}$ is a sequence such that $\Psi_{X,\epsilon}$ is convergent. $\Psi_{X,\epsilon}$ is a measure or a semi-positive (n, n) -current on X . Then from the construction of Ω_{can} in the proof of Theorem B.2 (also Theorem B.1)

$$\frac{\Psi_{X,\epsilon}}{\Omega_{can}} < \infty. \quad (1.7)$$

If the canonical ring $R(X, K_X)$ is finitely generated, one can replace $\Psi_{X,\epsilon}$ by

$$\Psi_X = \sum_{m=0}^M \sum_{j=0}^{d_m} |\sigma_{m,j}|^{\frac{2}{m}}$$

for some M sufficiently large.

Recently, the finite generation of canonical rings on algebraic varieties of general type was proved independently by [BiCaHaMc] and [Si2]. By assuming the finite generation of canonical rings, Theorem B.1 and B.2 can be strengthened and the proof can be very much simplified. It turns out that the canonical measure in Theorem B.1 and B.2 is unique and invariant under birational transformations.

Theorem C.1 *Let X be an algebraic manifold of general type. If the canonical ring $R(X, K_X)$ is finitely generated, the Kähler-Einstein measure constructed in Theorem B.1 is continuous on X and smooth on a Zariski open set of X . Furthermore, it is the pullback of the unique canonical Kähler-Einstein measure Ω_{KE} from X_{can} satisfying*

$$(\sqrt{-1} \partial \bar{\partial} \log \Omega_{KE})^n = \Omega_{KE}. \quad (1.8)$$

The unique Kähler-Einstein metric with continuous local potential and the associated Kähler-Einstein measure on the canonical model X_{can} is constructed in [EyGuZe1]. The Kähler-Einstein measure in Theorem C.1 is invariant under birational transformations and so it can be considered as a birational invariant. Theorem C.1 can also be generalized to all algebraic manifolds of positive Kodaira dimension.

Theorem C.2 *Let X be an n -dimensional algebraic manifold of Kodaira dimension $0 < \kappa < n$. If the canonical ring $R(X, K_X)$ is finitely generated, then there exists a unique canonical measure Ω_{can} on X satisfying*

1. $0 < \frac{\Psi_X}{\Omega_{can}} < \infty$.
2. Ω_{can} is continuous on X and smooth on a Zariski open set of X .
3. Let $\Phi : X \dashrightarrow X_{can}$ be the pluricanonical map. Then there exists a unique closed positive $(1, 1)$ -current ω_{can} with bounded local potential on X_{can} such that $\Phi^* \omega_{can} = \sqrt{-1} \partial \bar{\partial} \log \Omega_{can}$ outside the base locus of the pluricanonical system. Furthermore, on X_{can}

$$(\omega_{can})^\kappa = \Phi_* \Omega_{can},$$

and

$$\text{Ric}(\omega_{can}) = -\omega_{can} + \bar{\omega}_{WP}.$$

In particular, Ω_{can} is invariant under birational transformations.

$\bar{\omega}_{WP}$ is defined in Section 4.5 (cf. Definition 4.2) and it coincides with ω_{WP} in Theorem B.2 on a Zariski open set of X_{can} by choosing Y^\dagger to be X_{can} . Theorem C.1 and Theorem 3.2 are proved in Section 4.5.

2 Preliminaries

2.1 Kodaira dimension and semi-ample fibrations

Let X be an n -dimensional compact complex algebraic manifold and $L \rightarrow X$ a holomorphic line bundle over X . Let $N(L)$ be the semi-group defined by

$$N(L) = \{m \in \mathbf{N} \mid H^0(X, L^m) \neq 0\}.$$

Given any $m \in N(L)$, the linear system $|L^m| = \mathbf{P}H^0(X, L^m)$ induces a rational map ϕ_m

$$\Phi_m : X \dashrightarrow \mathbf{CP}^{d_m}$$

by any basis $\{\sigma_{m,0}, \sigma_{m,1}, \dots, \sigma_{m,d_m}\}$ of $H^0(X, L^m)$

$$\Phi_m(z) = [\sigma_{m,0}, \sigma_{m,1}, \dots, \sigma_{m,d_m}(z)],$$

where $d_m + 1 = \dim H^0(X, L^m)$. Let $Y_m = \Phi_m(X) \subset \mathbf{CP}^{d_m}$ be the image of the closure of the graph of Φ_m .

Definition 2.1 *The Iitaka dimension of L is defined to be*

$$\kappa(X, L) = \max_{m \in N(L)} \{\dim Y_m\}$$

if $N(L) \neq \emptyset$, and $\kappa(X, L) = -\infty$ if $N(L) = \emptyset$.

Definition 2.2 *Let X be an algebraic manifold and K_X the canonical line bundle over X . Then the Kodaira dimension $\text{kod}(X)$ of X is defined to be*

$$\text{kod}(X) = \kappa(X, K_X).$$

The Kodaira dimension is a birational invariant of an algebraic variety and the Kodaira dimension of a singular variety is equal to that of its smooth model.

Definition 2.3 *Let $L \rightarrow X$ be a holomorphic line bundle over a compact algebraic manifold X . L is called semi-ample if L^m is globally generated for some $m > 0$.*

For any $m \in \mathbb{N}$ such that L^m is globally generated, the linear system $|L^m|$ induces a holomorphic map Φ_m

$$\Phi_m : X \rightarrow \mathbb{CP}^{d_m}$$

by any basis of $H^0(X, L^m)$. Let $Y_m = \Phi_m(X)$ and so Φ_m can be considered as

$$\Phi_m : X \rightarrow Y_m.$$

The following theorem is well-known (cf. [La, Ue]).

Theorem 2.1 *Let $L \rightarrow X$ be a semi-ample line bundle over an algebraic manifold X . Then there is an algebraic fibre space*

$$\Phi_\infty : X \rightarrow Y$$

such that for any sufficiently large integer m with L^m being globally generated,

$$Y_m = Y \quad \text{and} \quad \Phi_m = \Phi_\infty,$$

*where Y is a normal algebraic variety. Furthermore, there exists an ample line bundle A on Y such that $L^m = (\Phi_\infty)^*A$.*

If L is semi-ample, the graded ring $R(X, L) = \bigoplus_{m \geq 0} H^0(X, L^m)$ is finitely generated and so $R(X, L) = \bigoplus_{m \geq 0} H^0(X, L^m)$ is the coordinate ring of Y .

Definition 2.4 *Let $L \rightarrow X$ be a semi-ample line bundle over an algebraic manifold X . Then the algebraic fibre space $\Phi_\infty : X \rightarrow Y$ as in Theorem 2.1 is called the Iitaka fibration associated to L and it is completely determined by the linear system $|L^m|$ for sufficiently large m .*

In particular, if the canonical bundle K_X is semi-ample, the algebraic fibre space associated to K_X

$$f : X \rightarrow X_{\text{can}}$$

is called the Iitaka fibration of X , where $f = \Phi_\infty$ and X_{can} is called the canonical model of X .

2.2 Iitaka fibrations

In general, the canonical line bundle is not necessarily semi-ample, and the asymptotic behavior of the pluricanonical maps is characterized by the following fundamental theorem on Kodaira dimensions due to Iitaka (cf. [Ue]).

Theorem 2.2 *Let X be an n -dimensional algebraic manifold of positive Kodaira dimension. Then for all sufficiently large $m \in N(K_X)$, the pluricanonical maps $\Phi_m : X \rightarrow Y_m$ are birationally equivalent to an algebraic fibre space*

$$\Phi^\dagger : X^\dagger \rightarrow Y^\dagger$$

unique up to birational equivalence satisfying

1. There exists a commutative diagram for sufficiently large $m \in N(K_X)$

$$\begin{array}{ccc} X & \xleftarrow{\pi^\dagger} & X^\dagger \\ \vdots \Phi_m \downarrow & & \downarrow \Phi^\dagger \\ Y_m & \xleftarrow{\mu_m} & Y^\dagger \end{array} \quad (2.1)$$

of rational maps with π^\dagger and μ_m being birational.

2. $\dim Y^\dagger = \text{kod}(X)$.
3. A very general fibre of Φ^\dagger has Kodaira dimension 0.

2.3 Analytic Zariski decomposition

Let X be a compact complex manifold and L be a holomorphic line bundle on X equipped with a smooth hermitian metric h_0 .

A singular hermitian metric h on L is given by

$$h = h_0 e^{-\varphi}$$

for some $\varphi \in L^1(M)$.

Let Θ_{h_0} be the curvature of h_0 defined by

$$\Theta_{h_0} = -\sqrt{-1} \partial \bar{\partial} \log h_0.$$

Then the curvature Θ_h of h as a current is defined by

$$\Theta_h = \Theta_{h_0} + \sqrt{-1} \partial \bar{\partial} \varphi.$$

Definition 2.5 *L is called pseudoeffective if there exists a singular hermitian metric h on L such that the curvature Θ_h is a closed positive current. Let*

$$\mathcal{P}_{h_0}(X) = \{\varphi \in L^1(X) \mid \Theta_{h_0} + \sqrt{-1} \partial \bar{\partial} \varphi \geq 0 \text{ as current}\}.$$

Definition 2.6 Let $\varphi \in \mathcal{P}_{h_0}(X)$ and $h = h_0 e^{-\varphi}$. The multiplier ideal sheaf $\mathcal{I}(h) \subset \mathcal{O}_X(L)$ or $\mathcal{I}(\varphi)$ is defined by

$$\Gamma(U, \mathcal{I}(h)) = \{f \in \Gamma(U, \mathcal{O}_X(L)) \mid |f|_{h_0}^2 e^{-\varphi} \in L_{loc}^1(U)\}.$$

The notion of analytic Zariski decomposition is an analytic analog of Zariski decomposition and it is introduced in [Ts1] to study a pseudoeffective line bundle.

Definition 2.7 A singular hermitian metric h on L is an analytic Zariski decomposition if

1. Θ_h is a closed semi-positive current,
2. for every $m \geq 0$, the natural inclusion

$$H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^m)) \rightarrow H^0(X, \mathcal{O}_X(mL))$$

is an isomorphism.

2.4 Complex Monge-Ampère equations

Let X be an n -dimensional Kähler manifold and let ω be a smooth closed semi-positive $(1, 1)$ -form. ω is Kähler if it is positive and ω is called big if $[\omega]^n = \int_X \omega^n > 0$.

Definition 2.8 A quasi-plurisubharmonic function associated to ω is a function $\varphi : X \rightarrow [-\infty, \infty)$ such that for any smooth local potential ψ of ω , $\psi + \varphi$ is plurisubharmonic. We denote by $PSH(X, \omega)$ the set of all quasi-plurisubharmonic functions associated to ω on X .

The following comparison principle for quasi-plurisubharmonic functions on compact Kähler manifolds is well-known.

Theorem 2.3 Let X be an n -dimensional Kähler manifold. Suppose $\varphi, \psi \in PSH(X, \omega)$ for a big smooth closed semi-positive $(1, 1)$ -form ω . Then

$$\int_{\varphi < \psi} (\omega + \sqrt{-1} \partial \bar{\partial} \psi)^n \leq \int_{\varphi < \psi} (\omega + \sqrt{-1} \partial \bar{\partial} \phi)^n.$$

In [Kol1], Kolodziej proved the fundamental theorem on the existence of continuous solutions to the Monge-Ampère equation $(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = F \omega^n$, where ω is a Kähler form and $F \in L^p(X, \omega^n)$ for some $p > 1$. Its generalization was independently carried out in [Zh] and [EyGuZe1]. They proved that there is a bounded solution when ω is semi-positive and big. A detailed proof for the continuity of the solution was given in [DiZh] (also see [Zh] for an earlier and sketched proof). These generalizations are summarized in the following.

Theorem 2.4 Let X be an n -dimensional Kähler manifold and let ω be a big smooth closed semi-positive $(1, 1)$ -form. Then there exists a unique continuous solution to the following Monge-Ampère equation

$$(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = F \Omega,$$

where $\Omega > 0$ is a smooth volume form on X , $F \in L^p(X, \Omega)$ for some $p > 1$ and $\int_X F \Omega = \int_X \omega^n$.

Recently, Demailly and Pali proved the following uniform estimate and we refer the readers to the general statement in [DePa]. Such an L^∞ -estimate is also independently obtained in [EyGuZe2].

Theorem 2.5 *Let X be an n -dimensional Kähler manifold. Let $\Omega > 0$ be a smooth volume form and ω be a smooth Kähler form such that $\omega \leq \omega_0$ for some smooth Kähler form ω_0 on X . Let $\varphi \in PSH(X, \omega) \cap L^\infty(X)$ be a solution of the degenerate complex Monge-Ampère equation $(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = F\Omega$ with $F \in L^p(X)$ for some $p > 1$. Suppose*

1. $\int_X \left(\frac{F}{[\omega]^n} \right)^p \Omega \leq A,$
2. $\frac{\omega^n}{[\omega]^n \Omega} + \int_X \left(\frac{[\omega]^n \Omega}{\omega^n} \right)^\epsilon \Omega \leq B, \text{ for some } \epsilon > 0.$

Then

$$\sup_X \varphi - \inf_X \varphi \leq C(\Omega, \omega_0, \epsilon, p, A, B).$$

The estimate in Theorem 2.5 assumes very weak dependence on the reference form ω and it is essential in deriving the C^0 -estimate for the Kähler-Ricci flow on algebraic manifolds with positive Kodaira dimension and semi-ample canonical line bundle (cf. Section 5.2).

3 Canonical metrics for semi-ample canonical bundle

3.1 Canonical metrics on canonical models

Let X be an n -dimensional complex algebraic manifold with semi-ample canonical line bundle K_X . Fix $m \in N(K_X)$ sufficiently large and let $f = \Phi_m$, the Iitaka fibration of X is then given by the following holomorphic map

$$f : X \rightarrow X_{can} \subset \mathbf{CP}^{d_m}.$$

We assume that $0 < \kappa = \text{kod}(X) < n$ and so X is an algebraic fibre space over X_{can} . Let

$$X_{can}^\circ = \{y \in X_{can} \mid y \text{ is a nonsingular and } X_y = f^{-1}(y) \text{ is nonsingular fibre}\}$$

and $X^\circ = f^{-1}(X_{can}^\circ)$. The following proposition is well-known.

Proposition 3.1 *We have*

$$K_X = \frac{1}{m} f^* \mathcal{O}(1).$$

For all $y \in X_{can}^\circ$, K_{X_y} is numerically trivial and so $c_1(X_y) = 0$.

Thus X can be considered as a holomorphic fibration of polarized Calabi-Yau manifolds over its canonical model X_{can} . Since $f : X \rightarrow X_{can} \in \mathbf{CP}^{d_m}$ and $-c_1(X) = \frac{1}{m} [f^* \mathcal{O}(1)]$, we can define

$$\chi = \frac{1}{m} \sqrt{-1} \partial \bar{\partial} \log \sum_{j=0}^{d_m} |\sigma_{m,j}|^2 \in -2\pi c_1(X)$$

as a multiple of the pulled back Fubini-Study metric on \mathbf{CP}^{d_m} by a basis $\{\sigma_{m,j}\}_{j=0}^{d_m} \subset H^0(X, K_X^m)$.

We can also consider χ as the restriction of the multiple of the Fubini-Study metric on the normal variety X_{can} and we identify χ and $f^*\chi$ for convenience.

Let $\Omega = \sum_{j=0}^{d_m} |\sigma_{m,j}|^2$. Since K_X^m is base point free, Ω is a smooth nondegenerate volume form on X such that

$$\sqrt{-1}\partial\bar{\partial}\log\Omega = \chi.$$

Definition 3.1 *The pushforward $f_*\Omega$ with respect to the holomorphic map $f : X \rightarrow X_{can}$ is defined as currents as the following. For any continuous function ψ on X_{can}*

$$\int_{X_{can}} \psi f_*\Omega = \int_X (f^*\psi)\Omega.$$

Lemma 3.1 *On X_{can}° ,*

$$f_*\Omega = \int_{X_y} \Omega.$$

Definition 3.2 *We define a function F on X_{can} by*

$$F = \frac{f_*\Omega}{\chi^\kappa}. \quad (3.1)$$

Lemma 3.2 *Given any Kähler class $[\omega]$ on X , there is a smooth function ψ on X° such that $\omega_{SF} := \omega + \sqrt{-1}\partial\bar{\partial}\psi$ is a closed semi-flat $(1,1)$ -form in the following sense: the restriction of ω_{SF} to each smooth $X_y \subset X^\circ$ is a Ricci flat Kähler metric.*

Proof For each $y \in X_{can}^\circ$, let ω_y be the restriction of ω to X_y and ∂_V and $\bar{\partial}_V$ be the restriction of ∂ and $\bar{\partial}$ to X_y . Then by the Hodge theory, there is a unique function h_y on X_y defined by

$$\begin{cases} \partial_V \bar{\partial}_V h_y = -\partial_V \bar{\partial}_V \log \omega_y^{n-\kappa}, \\ \int_{X_y} e^{h_y} \omega_y^{n-\kappa} = \int_{X_y} \omega_y^{n-\kappa}. \end{cases} \quad (3.2)$$

By Yau's solution to the Calabi conjecture, there is a unique ψ_y solving the following Monge-Ampère equation

$$\begin{cases} \frac{(\omega_y + \sqrt{-1}\partial_V \bar{\partial}_V \psi_y)^{n-\kappa}}{\omega_y^{n-\kappa}} = e^{h_y} \\ \int_{X_y} \psi_y \omega_y^{n-\kappa} = 0. \end{cases} \quad (3.3)$$

Since f is holomorphic, $\psi(z, s) = \psi_y(z)$ is well-defined as a smooth function on X° . □

For each $y \in X_{can}^\circ$, there exists a holomorphic $(n - \kappa, 0)$ form η on X_y such that $\eta \wedge \bar{\eta}$ is a Calabi-Yau volume form and $\int_{X_y} \eta \wedge \bar{\eta} = \int_{X_y} (\omega|_{X_y})^{n-\kappa}$.

Definition 3.3 The closed $(n - \kappa, n - \kappa)$ -current Θ on X° is defined to be

$$\Theta = (\omega_{SF})^{n-\kappa}. \quad (3.4)$$

Let $\Theta_y = \Theta|_{X_y}$ for $y \in X_{can}^\circ$ be the restriction of Θ on a nonsingular fibre X_y . Then Θ_y is a smooth Calabi-Yau volume form with

$$\int_{X_y} \Theta_y = [\omega]^{n-\kappa} \cdot X_y = \text{constant}.$$

We can always scale $[\omega]$ such that $[\omega]^{n-\kappa} \cdot X_y = 1$ for $y \in X_{can}^\circ$.

Lemma 3.3 On X° , we have

$$f^*F = \left(\frac{\Omega}{\Theta \wedge \chi^\kappa} \right). \quad (3.5)$$

Furthermore, Θ can be extended to X as current such that $f^*F = \left(\frac{\Omega}{\Theta \wedge \chi^\kappa} \right)$ on X .

Proof Let $\mathcal{F} = \left(\frac{\Omega}{\Theta \wedge \chi^\kappa} \right)$ be defined on X° . We first show that \mathcal{F} is constant along each fibre X_y for $y \in X_{can}^\circ$.

Since χ is the pullback from X_{can} , we have

$$\sqrt{-1}\partial_V\bar{\partial}_V \log \Omega = \sqrt{-1}\partial_V\bar{\partial}_V \log \Theta \wedge \chi^\kappa = 0$$

on each nonsingular fibre X_y . On the other hand, \mathcal{F} is smooth on each X_y for $y \in X_{can}^\circ$, therefore \mathcal{F} is constant along X_y and \mathcal{F} can be considered as the pullback of a function from X_{can}° .

Now we can show (3.5). Let ψ be any smooth test function on X_{can}° . Let y_0 be a fixed point in X_{can}° .

$$\int_{X_{can}^\circ} \psi F \chi^\kappa = \int_{X_{can}^\circ} \psi f_* \Omega = \int_{y \in X_{can}^\circ} \psi \left(\int_{X_y} \Omega \right) = \int_{X^\circ} \psi \Omega$$

On the other hand,

$$\begin{aligned} \int_{X_{can}^\circ} \psi \mathcal{F} \chi^\kappa &= \int_{y \in X_{can}^\circ} \psi \left(\frac{\Omega}{\Theta \wedge \chi^\kappa} \right) \left(\int_{X_y} \Theta_y \right) \chi^\kappa \\ &= \int_{y \in X_{can}^\circ} \int_{X_y} \psi \left(\frac{\Omega}{\Theta \wedge \chi^\kappa} \right) \Theta_y \wedge \chi^\kappa \\ &= \int_{X^\circ} \psi \Omega. \end{aligned}$$

Therefore $f^*F = \mathcal{F}$.

Let ψ' be any smooth test function on X . Θ can be extended as current to X such that

$$\int_X \psi' f^* F \Theta \wedge \chi^\kappa = \int_X \psi' \Omega.$$

□

Proposition 3.2 *F is smooth on X_{can}° and there exists $\epsilon > 0$ such that*

$$F \in L^{1+\epsilon}(X_{can}).$$

Proof Calculate

$$\int_{X_{can}} F^{1+\epsilon} \chi^\kappa = \int_X (f^* F)^{1+\epsilon} \chi^\kappa \wedge \Theta = \int_X (f^* F)^\epsilon \Omega.$$

Also for any $y \in X_{can}^\circ$, we can choose $z_0 \in X_y$ such that on X_y $\omega^{n-\kappa}(z_0) = \omega_{SF}^{n-\kappa}(z_0)$ since X_y is smooth and $\int_{X_y} \omega^{n-\kappa} = \int_{X_y} \omega_{SF}^{n-\kappa}$. Then

$$\begin{aligned} |F(y)| &= \frac{\Omega}{\chi^\kappa \wedge \omega_{SF}^{n-\kappa}} \\ &= \frac{\Omega}{\chi^\kappa \wedge \omega^{n-\kappa}} \frac{\omega^{n-\kappa}}{\omega_{SF}^{n-\kappa}}(z_0) \\ &= \frac{\Omega}{\chi^\kappa \wedge \omega^{n-\kappa}}(z_0) \\ &\leq \sup_{X_y} \frac{\Omega}{\chi^\kappa \wedge \omega^{n-\kappa}}. \end{aligned}$$

Therefore F is bounded by poles and $(f^* F)^\epsilon$ is integrable for sufficiently small $\epsilon > 0$.

□

Proposition 3.3 *Let $\pi : Y \rightarrow X_{can}$ be a smooth model of X_{can} by resolution of singularities of X_{can} . $\pi^* F$ has at worst pole singularities on Y .*

Proof Let D be a divisor on X_{can} such that $X_{can} \setminus X_{can}^\circ \subset S_D$. Let S_D be the defining section of D and h_D be the hermitian metric on the line bundle associated to $[D]$ such that $\pi^* (|S_D|_{h_D}^2)$ is a smooth function. For any continuous volume form Ω' on X ,

$$\begin{aligned} &f_* (|S_D|_{h_D}^{2N} \Omega') \\ &= \int_{X_s} |S_D|_{h_D}^{2N} \Omega' \\ &= \left(\int_{X_s} |S_D|_{h_D}^{2N} \left(\frac{\Omega'}{\chi^\kappa \wedge \omega^{n-\kappa}} \right) (\omega|_{X_s})^{n-\kappa} \right) \chi^\kappa \end{aligned}$$

for $s \in X_{can}^\circ$. Since $|S_D|_{h_D}^{2N} \frac{\Omega'}{\chi^\kappa \wedge \omega^{n-\kappa}} < \infty$ for sufficiently large N , there exists a constant C such that

$$0 \leq f_* (|S_D|_{h_D}^{2N} \Omega') < \chi^\kappa.$$

Let $\mathcal{F}_N = |S_D|_{h_D}^{2N} F$. Then on X_{can}°

$$\begin{aligned}
& \sqrt{-1} \partial \bar{\partial} \mathcal{F}_N \\
&= \sqrt{-1} \partial \bar{\partial} \left(\int_{X_s} |S_D|_{h_D}^{2N} \frac{\Omega}{\chi^\kappa \wedge \omega^{n-\kappa}} \omega^{n-\kappa} \right) \\
&= \int_{X_s} \sqrt{-1} \partial \bar{\partial} \left(|S_D|_{h_D}^{2N} \frac{\Omega}{\chi^\kappa \wedge \omega^{n-\kappa}} \right) \wedge \omega^{n-\kappa} \\
&\leq C |S_D|_{h_D}^{2M} \int_{X_s} \omega^{n-\kappa+1}
\end{aligned}$$

for some sufficiently large M by choosing N sufficiently large.

Let η be any semi-positive smooth $(\kappa-1, \kappa-1)$ -form supported on X_{can}° . Then

$$\begin{aligned}
& \int_{X_{can}} \eta \wedge \sqrt{-1} \partial \bar{\partial} \mathcal{F}_N \\
&\leq C |S_D|_{h_D}^{2M} \int_X f^* \eta \wedge \omega^{n-\kappa+1} \\
&= C |S_D|_{h_D}^{2M} \int_{X_{can}} \left(\frac{\int_{X_s} \omega^{n-\kappa+1} \wedge \chi^{\kappa-1}}{\chi^\kappa} \right) f^* \eta \wedge \chi \\
&\leq C' |S_D|_{h_D}^{2L} \int_{X_{can}} \eta \wedge \chi
\end{aligned}$$

if we chose N sufficiently large.

Similar lower bound of $\sqrt{-1} \partial \bar{\partial} \mathcal{F}_N$ can be achieved and so for sufficiently large N , on X_{can}°

$$-C\chi \leq \sqrt{-1} \partial \bar{\partial} \mathcal{F}_N \leq C\chi.$$

Let ω_Y be a Kähler metric on Y and Δ_Y be the Laplace operator associated to ω_Y . Then for sufficiently large N

$$|\Delta_Y \pi^* \mathcal{F}_N| \leq C.$$

Also we can assume that $\pi^* (|S_D|_{h_D}^{2N}) \omega_Y \leq f^* \chi$ for sufficiently large N . After repeating the above estimates, we have for any $k \geq 0$, there exists sufficiently large N and $C_{k,N}$ such that

$$\left| (\Delta_Y)^k (\pi^* \mathcal{F}_N) \right| \leq C_{k,N}.$$

By standard elliptic estimates, $\pi^* \mathcal{F}_N$ is uniformly bounded in C^k if we choose N sufficiently large, therefore $\pi^* F$ can have at worst pole singularities. □

Now let us recall some facts on the Weil-Petersson metric on the moduli space \mathcal{M} of polarized Calabi-Yau manifolds of dimension $n-\kappa$. Let $\mathcal{X} \rightarrow \mathcal{M}$ be a universal family of Calabi-Yau

manifolds. Let $(U; t_1, \dots, t_\kappa)$ be a local holomorphic coordinate chart of \mathcal{M} , where $\kappa = \dim \mathcal{M}$. Then each $\frac{\partial}{\partial t_i}$ corresponds to an element $\iota(\frac{\partial}{\partial t_i}) \in H^1(\mathcal{X}_t, T_{\mathcal{X}_t})$ through the Kodaira-Spencer map ι . The Weil-Petersson metric is defined by the L^2 -inner product of harmonic forms representing classes in $H^1(\mathcal{X}_t, T_{\mathcal{X}_t})$. In the case of Calabi-Yau manifolds, as first shown in [Ti4], it can be expressed as follows. Let Ψ be a nonzero holomorphic $(n - \kappa, 0)$ -form on the fibre \mathcal{X}_t and $\Psi \lrcorner \iota(\frac{\partial}{\partial t_i})$ be the contraction of Ψ and $\frac{\partial}{\partial t_i}$. Then the Weil-Petersson metric is given by

$$\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \right)_{\omega_{WP}} = \frac{\int_{\mathcal{X}_t} \Psi \lrcorner \iota(\frac{\partial}{\partial t_i}) \wedge \overline{\Psi \lrcorner \iota(\frac{\partial}{\partial t_j})}}{\int_{\mathcal{X}_t} \Psi \wedge \overline{\Psi}}. \quad (3.6)$$

One can also represent ω_{WP} as the curvature form of the first Hodge bundle $f_*\Omega_{\mathcal{X}/\mathcal{M}}^{n-\kappa}$ (cf. [Ti4]). Let Ψ be a nonzero local holomorphic section of $f_*\Omega_{\mathcal{X}/\mathcal{M}}^{n-\kappa}$ and one can define the hermitian metric h_{WP} on $f_*\Omega_{\mathcal{X}/\mathcal{M}}^{n-\kappa}$ by

$$|\Psi_t|_{h_{WP}}^2 = \int_{\mathcal{X}_t} \Psi_t \wedge \overline{\Psi_t}. \quad (3.7)$$

Then the Weil-Petersson metric is given by

$$\omega_{WP} = \text{Ric}(h_{WP}). \quad (3.8)$$

The Weil-Petersson metric can also be considered a canonical hermitian metric on the dualizing sheaf $f_*(\Omega_{X/X_{can}}^{n-\kappa}) = (f_{*1}\mathcal{O}_X)^\vee$ over X_{can}° .

Let X be an n -dimensional algebraic manifold. Suppose its canonical line bundle K_X is semi-positive and $0 < \kappa = \text{kod}(X) < n$. Let X_{can} be the canonical model of X . We define a canonical hermitian metric h_{can} on $f_*(\Omega_{X/X_{can}}^{n-\kappa})$ in the way that for any smooth $(n - \kappa, 0)$ -form η on a nonsingular fibre X_y ,

$$|\eta|_{h_{WP}}^2 = \frac{\eta \wedge \bar{\eta} \wedge \chi^\kappa}{\Theta \wedge \chi^\kappa} = \frac{\int_{X_y} \eta \wedge \bar{\eta}}{\int_{X_y} \Theta}. \quad (3.9)$$

Definition 3.4 *Let X be an n -dimensional algebraic manifold with semi-ample canonical line bundle K_X . Suppose $0 < \text{kod}(X) < n$ and so $f : X \rightarrow X_{can}$ is a holomorphic fibration of Calabi-Yau manifolds. A closed positive $(1, 1)$ -current ω on X_{can} is called a canonical metric if it satisfies the following.*

1. $f^*\omega \in -2\pi c_1(X)$.
2. ω is smooth outside a subvariety of X_{can} and $(f^*\omega)^\kappa \wedge \Theta$ is continuous on X .
3. $\text{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log \omega^\kappa$ is well-defined on X as a current and on X_{can}°

$$\text{Ric}(\omega) = -\omega + \omega_{WP}. \quad (3.10)$$

Definition 3.5 *Suppose ω_{can} is a canonical metric on X_{can} . We define the canonical volume form Ω_{can} on X to be*

$$\Omega_{KE} = (f^*\omega_{KE})^\kappa \wedge \Theta. \quad (3.11)$$

3.2 Existence and uniqueness

The main goal of this section is to prove the existence and uniqueness for canonical metrics on the canonical models.

Theorem 3.1 *Let X be an n -dimensional algebraic manifold with semi-ample canonical line bundle K_X . Suppose $0 < \kappa = \text{kod}(X) < n$. There exists a unique canonical metric on X_{can} .*

We will need the following theorem of solving singular Monge-Ampère equation to prove Theorem 3.1.

Theorem 3.2 *There exists a unique solution $\varphi \in PSH(X) \cap C^0(X_{\text{can}}) \cap C^\infty(X_{\text{can}}^\circ)$ to the following Monge-Ampère equation on X_{can}*

$$(\chi + \sqrt{-1}\partial\bar{\partial}\varphi)^\kappa = Fe^\varphi\chi^\kappa. \quad (3.12)$$

Proof of Theorem 3.1 We will prove Theorem 3.1 by assuming Theorem 3.2.

Let φ be the solution in Theorem 3.1 and $\omega = \chi + \sqrt{-1}\partial\bar{\partial}\varphi$.

1.

$$f^*\omega = f^*\chi + \sqrt{-1}\partial\bar{\partial}f^*\varphi \in -c_1(X).$$

and it proves 1. in Definition 3.4.

2. By Theorem 3.2, ω is smooth on X_{can}° and

$$(f^*\omega)^\kappa \wedge \Theta = \Omega e^{f^*\varphi}$$

is continuous since $f^*\varphi$ is continuous on X and Ω is a smooth volume form. This proves 2. in Definition 3.4

3. Then

$$\text{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log \omega^\kappa = -\sqrt{-1}\partial\bar{\partial}\log \chi^\kappa - \sqrt{-1}\partial\bar{\partial}\log F - \sqrt{-1}\partial\bar{\partial}\varphi$$

is well-defined as a current on X_{can} .

Calculate on X_{can}°

$$\begin{aligned} & \sqrt{-1}\partial\bar{\partial}\log \chi^\kappa + \sqrt{-1}\partial\bar{\partial}\log F + \sqrt{-1}\partial\bar{\partial}\varphi \\ = & \sqrt{-1}\partial\bar{\partial}\log \chi^\kappa + \sqrt{-1}\partial\bar{\partial}\log \left(\frac{\Omega}{\Theta \wedge \chi^\kappa} \right) + \omega - \chi \\ = & \omega + (-\sqrt{-1}\partial\bar{\partial}\log (\Theta \wedge \chi^\kappa) + \sqrt{-1}\partial\bar{\partial}\log \chi^\kappa) \\ = & \omega - \omega_{WP}. \end{aligned}$$

Therefore

$$\text{Ric}(\omega) = -\omega + \omega_{WP}.$$

So we have proved 3. in Definition 3.4.

Now we will prove the uniqueness of the canonical metric.

Let $\omega = \chi + \sqrt{-1}\partial\bar{\partial}\varphi$ be a canonical metric on X_{can} . Then by the equation for the canonical metric, we have on X_{can}°

$$\sqrt{-1}\partial\bar{\partial}\log\left(\frac{\omega^\kappa}{\chi^\kappa}\right) = \sqrt{-1}\partial\bar{\partial}\log\left(\frac{\Omega}{\Theta \wedge \chi^\kappa}e^\varphi\right).$$

Let $\xi = \left(\frac{\omega^\kappa}{\chi^\kappa}\right)\left(\frac{\Omega}{\Theta \wedge \chi^\kappa}e^\varphi\right)^{-1}$. Then on X_{can}° we have

$$\sqrt{-1}\partial\bar{\partial}\log\xi = 0.$$

On the other hand,

$$f^*\xi = \frac{\omega^\kappa \wedge \Theta}{\Omega}e^{-\varphi}$$

extends to a strictly positive continuous function on X . Since f is a holomorphic map, ξ extends to a continuous function on X_{can} . Let $\pi : X'_{can} \rightarrow X_{can}$ be a resolution of X_{can} . Then $\pi^*\xi$ is continuous on X'_{can} and $\sqrt{-1}\partial\bar{\partial}\log\pi^*\xi = 0$ so that $\pi^*\xi = \text{constant}$ on X'_{can} . Therefore $\xi = \text{constant} > 0$ on X_{can} and $\varphi' = \varphi + \log\xi \in PSH(\chi) \cap C^0(X_{can})$ solves the Monge-Ampère equation (3.12). The uniqueness of the solution for (3.12) implies the uniqueness of the canonical metric. \square

Corollary 3.1 *Let ω_{can} be a canonical metric on X_{can} and Ω_{can} the canonical volume form on X . Then*

$$f^*\omega_{can} = \sqrt{-1}\partial\bar{\partial}\log\Omega_{can}. \quad (3.13)$$

Proof Let $\omega_{can} = \chi + \sqrt{-1}\partial\bar{\partial}\varphi$, where φ be the solution of the Monge-Ampère equation (3.12) in Theorem 3.2. Then

$$\Omega_{can} = (f^*\omega_{can})^\kappa \wedge \Theta = \Omega e^{f^*\varphi}$$

and so

$$\sqrt{-1}\partial\bar{\partial}\log\Omega_{can} = \sqrt{-1}\partial\bar{\partial}\log\Omega + \sqrt{-1}\partial\bar{\partial}f^*\varphi = f^*\omega_{can}. \quad \square$$

Proof of Theorem 3.2

Step 1. Approximation

Let $\pi : Y \rightarrow X_{can}$ be a resolution of singularities such that $E = \pi^*(X_{can} \setminus X_{can}^\circ)$ is a divisor with simple normal crossings. Let $\hat{\chi} = \pi^*\chi$ and $\hat{F} = \pi^*F$. Then $\hat{\chi}$ is a closed semi-positive $(1,1)$ -form on Y and $\int_Y \hat{\chi}^\kappa = \int_{X_{can}} \chi^\kappa > 0$, i.e., $\hat{\chi}$ is big. We will consider the following Monge-Ampère equation on Y

$$\frac{(\hat{\chi} + \sqrt{-1}\partial\bar{\partial}\hat{\varphi})^\kappa}{\hat{\chi}^\kappa} = \hat{F}e^{\hat{\varphi}}. \quad (3.14)$$

Since $\hat{\chi}$ is a big semi-positive closed $(1, 1)$ -form and so $\hat{L} = \frac{1}{m}\pi^*\mathcal{O}(1)$ is a semi-positive big line bundle on Y . By Kodaira's lemma, there exists a divisor D such that for any $\epsilon > 0$, $[L] - \epsilon[D]$ is an ample divisor on Y . Let S_D be the defining section of D and choose a smooth hermitian metric h_D on the line bundle associated to $[D]$ such that

$$\hat{\chi} + \epsilon\sqrt{-1}\partial\bar{\partial}\log h_D > 0.$$

Fix $\epsilon_0 > 0$ and define a Kähler form $\omega_0 = \hat{\chi} + \epsilon_0\sqrt{-1}\partial\bar{\partial}\log h_D > 0$.

$\hat{F} \in L^{1+\epsilon}(Y, \hat{\chi}^\kappa)$ for some $\epsilon > 0$ since $F \in L^{1+\epsilon}(X_{can}, \chi^\kappa)$ for some $\epsilon > 0$. Then for each $k > 0$ there exists a family of functions $\{F_j\}_{j=1}^\infty$ on Y satisfying the following.

1. $F_j \in C^4(Y)$ for all j and $F_j \rightarrow \hat{F}$ in $L^{1+\epsilon}(Y, \hat{\chi}^\kappa)$ as $j \rightarrow \infty$.
2. There exists $C > 0$ such that $\log F_j \geq -C$ for all j .
3. There exist $\lambda, C > 0$ such that for all j

$$|||S_E|_{h_E}^{2\lambda} F_j|||_{C^2(Y)} \leq C,$$

where S_E is a defining section of E and h_E is a fixed smooth hermitian metric on the line bundle associated to $[E]$.

For example, we can choose \hat{F}_j to be defined by

$$F_j = \exp \left(\left(\frac{|S_E|_{h_E}^{2\alpha}}{j^{-1} + |S_E|_{h_E}^{2\alpha}} \right) \log \hat{F} \right)$$

for sufficiently large $\alpha > 0$.

We also choose a Kähler form ω_0 and let $\chi_j = \hat{\chi} + \frac{1}{j}\omega_0$. We consider the following Monge-Ampère equation

$$\frac{(\chi_j + \sqrt{-1}\partial\bar{\partial}\varphi_j)^\kappa}{(\chi_j)^\kappa} = F_j e^{\varphi_j} \quad (3.15)$$

for sufficiently large α .

By Yau's solution to the Calabi conjecture, for each j , there exists a unique solution $\varphi_j \in C^3(Y) \cap C^\infty(Y \setminus E)$ solving (3.15). We will derive uniform estimates for φ_j .

Step 2. Zeroth order estimates

Proposition 3.4 *There exists $C > 0$ such that for all j ,*

$$||\varphi_j||_{L^\infty(Y)} \leq C. \quad (3.16)$$

Furthermore, there exists a unique solution $\varphi_\infty \in PSH(Y, \hat{\chi}) \cap C^0(Y)$ solving the Monge-Ampère equation (3.14) such that

$$\varphi_j \rightarrow \varphi_\infty \quad (3.17)$$

in $L^1(Y, \omega_0^\kappa)$ as $j \rightarrow \infty$.

Proof By Yau's theorem, for each j , there exists a smooth solution φ_j solving the Monge-Ampère equation (3.14). We first derive a uniform upper bound for φ_j . Suppose that φ_j achieves its maximum at y_0 on Y . Then applying the maximum principle, we have

$$\varphi_j \leq \varphi_j(y_0) \leq \frac{1}{F_j(y_0)} \leq \sup_Y \left(\frac{1}{F_j} \right) \leq C.$$

Let Ω_0 be a smooth nowhere vanishing volume form on Y . We have to verify that $F_j \left(\frac{\chi_j^\kappa}{\Omega} \right)$ is uniformly bounded in $L^{1+\epsilon}(Y, \hat{\Omega})$ for some $\delta > 0$ for all j .

$$\left\| e^{\varphi_j} F_j \left(\frac{\chi_j^\kappa}{\Omega_0} \right) \right\|_{L^{1+\epsilon}(Y, \Omega_0)}^{1+\epsilon} = \int_Y e^{(1+\epsilon)\varphi_j} F_j^{1+\epsilon} \left(\frac{\chi_j^\kappa}{\Omega_0} \right)^\epsilon \chi_j^\kappa \leq C \int_Y F_j^{1+\epsilon} \chi_j^\kappa = C \|F_j\|_{L^{1+\epsilon}(Y, \Omega_0)}^{1+\epsilon}.$$

Therefore $\|\varphi_j\|_{L^\infty(Y)}$ is uniformly bounded by Theorem 2.4, since $\|F_j\|_{L^{1+\epsilon}(Y, \Omega_0)}^{1+\epsilon}$ is uniformly bounded.

Also Theorem 2.4 gives a unique solution $\varphi_\infty \in PSH(Y, \hat{\chi}) \cap C^0(Y)$ solving the Monge-Ampère equation 3.14. By the uniqueness of such φ_∞ , we have $\varphi_j \rightarrow \varphi_\infty$ in L^1 by standard potential theory. □

Step 3. Second order estimates

We now apply the maximum principle and prove a second order estimate for φ_j by using a modified argument in [Ya2]. Note that Tsuji used a similar trick in [Ts1] for the second-order estimate.

Let $\omega_j = \chi_j + \sqrt{-1}\partial\bar{\partial}\varphi_j$, Δ_0 and Δ_j be the Laplace operator associated to ω_0 and ω_j . The following lemma is proved by standard calculation.

Lemma 3.4 *There exists a uniform constant $C > 0$ only depending on ω_0 such that on $Y \setminus (E \cup D)$*

$$\Delta_j \log \text{tr}_{\omega_0}(\omega_j) \geq -C \left(1 + \text{tr}_{\omega_j}(\omega_0) + \frac{1 + |\Delta_0 \log F_j|}{\text{tr}_{\omega_0}(\omega_j)} \right).$$

Theorem 3.3 *There exist $\alpha, \beta, C > 0$ such that for all j and $z \in Y \setminus (E \cup D)$*

$$\text{tr}_{\omega_0}(\omega_j)(z) \leq \frac{C}{|S_E|_{h_E}^{2\alpha} |S_D|_{h_D}^{2\beta}} \quad (3.18)$$

Proof Define

$$\Phi_j = \varphi_j - \epsilon_0 \log |S_D|_{h_D}^2$$

and

$$H_j = \log (|S_E|_{h_E}^{2\alpha} \text{tr}_{\omega_0}(\omega_j)) - A\Phi_j$$

for some constants $\alpha, A > 0$ to be determined later.

First calculate on $Y \setminus (E \cup D)$

$$\begin{aligned}
& \Delta_j H_j \\
&= \Delta_j \log \operatorname{tr}_{\omega_0}(\omega_j) - A \Delta_j \Phi_j + \alpha \Delta_j \log |S_E|_{h_E}^2 \\
&= \Delta_j \log \operatorname{tr}_{\omega_0}(\omega_j) - A \operatorname{tr}_{\omega_j}(\omega_j - \omega_0) - \alpha \operatorname{tr}_{\omega_j}(\operatorname{Ric}(h_E)) \\
&\geq (A - C_1) \operatorname{tr}_{\omega_j}(\omega_0) - \alpha \operatorname{tr}_{\omega_j}(\operatorname{Ric}(h_E)) - \frac{C_1 (1 + |\Delta_0 \log F_j|)}{\operatorname{tr}_{\omega_0}(\omega_j)} - \frac{C_1}{\operatorname{tr}_{\omega_0}(\omega_j)} - \kappa A - C_1
\end{aligned}$$

Choose a sufficiently large $\alpha > 0$, such that there exists a constant $C_2 > 0$ with the following:

$$\frac{|S_E|_{h_E}^{2\alpha} (1 + |\Delta_0 \log F_j|)}{\operatorname{tr}_{\omega_0}(\omega_j)} \leq \frac{C_2}{\operatorname{tr}_{\omega_0}(\omega_j)}$$

and

$$|S_E|_{h_E}^{2\alpha} F_j \leq C_2.$$

Applying the elementary inequality

$$\begin{aligned}
\operatorname{tr}_{\omega_j}(\omega_0) &\geq C_3 (\operatorname{tr}_{\omega_0}(\omega_j))^{\frac{1}{\kappa-1}} \left(\frac{\omega_0^\kappa}{\omega_j^\kappa} \right)^{\frac{1}{\kappa-1}} \\
&= C_3 \left(\frac{e^{-\varphi_j}}{F_j} \operatorname{tr}_{\omega_0}(\omega_j) \left(\frac{\omega_0^\kappa}{\chi_j^\kappa} \right) \right)^{\frac{1}{\kappa-1}} \\
&\geq C_4 (|S_E|_{h_E}^{2\alpha} \operatorname{tr}_{\omega_0}(\omega_j))^{\frac{1}{\kappa-1}}
\end{aligned}$$

We can always choose A sufficiently large such that

$$\Delta_j H_j \geq C_5 A (|S_E|_{h_E}^{2\alpha} \operatorname{tr}_{\omega_0}(\omega_j))^{\frac{1}{\kappa-1}} - \frac{C_4 |S_E|_{h_E}^{4\alpha}}{|S_E|_{h_E}^{2\alpha} \operatorname{tr}_{\omega_0}(\omega_j)} - C_6 A.$$

For any j , suppose

$$\sup_{z \in Y} H_j(z) = H_j(z_0)$$

for some $z_0 \in Y \setminus (E \cup D)$ since $H_j = -\infty$ along $E \cup D$. By the maximum principle, $\Delta_j H_j(z_0) \leq 0$. By straightforward calculation, there exists $C_7 > 0$ such that

$$|S_E|_{h_E}^{2\alpha} \operatorname{tr}_{\omega_0}(\omega_j)|_{z=z_0} \leq C_7.$$

Hence there exists a uniform constant $C_8 > 0$ such that

$$H_j \leq H_j(z_0) = \log (|S_E|_{h_E}^{2\alpha} \operatorname{tr}_{\omega_0}(\omega_j))|_{z=z_0} - \varphi_j(z_0) + \epsilon_0 \log |S_D|_{h_D}^2(z = z_0) \leq C_8.$$

The theorem is proved since

$$\mathrm{tr}_{\omega_0}(\omega_j) \leq |S_E|_{h_E}^{-2\alpha} \exp(H_j + A\Phi_j) \leq \frac{C_9}{|S_E|_{h_E}^{2\alpha} |S_D|_{h_D}^{2\beta}}.$$

□

Step 4. C^k estimates

From the second order estimates, the Monge-Ampère equation (3.15) is uniformly elliptic on any compact set of $Y \setminus (E \cup D)$. The C^k -estimates are local estimates and can be derived by standard Schauder estimates and elliptic estimates. Therefore for any compact subset K of $Y \setminus (E \cup D)$, there exist constants $C_{K,R}$ such that

$$\|\varphi_j\|_{C^k(K)} \leq C_{K,R}.$$

Step 5. Proof of Theorem 3.2

By taking a sequence, we can assume $\varphi_j \rightarrow \varphi_\infty \in L^1(Y, \omega_0^k)$. Therefore

$$\varphi_\infty \in C^0(Y) \cap C^\infty(Y \setminus (E \cup D)).$$

On the other hand, one can choose different divisors D such that $[\hat{\chi}] - \epsilon[D] > 0$ for all sufficiently small $\epsilon > 0$ and the intersection of such divisors is contained in E . Therefore $\varphi_\infty \in C^\infty(Y \setminus E)$. Each fibre of the resolution π is connected and so φ_∞ is constant along the fibre since $\varphi_\infty \in PSH(Y, \hat{\chi}) \cap C^0(Y)$ and $\hat{\chi} \geq 0$. Therefore φ_∞ descends to a solution $\varphi \in PSH(X_{can}, \chi) \cap C^0(X_{can})$ solving equation (3.12) as in Theorem 3.2. Furthermore, φ is smooth on X_{can}° . This completes the proof of Theorem 3.2.

□

4 Canonical measures on algebraic manifolds of non-negative Kodaira dimension

4.1 Canonical measures on surfaces of non-negative Kodaira dimension

Let X be a Kähler surface of positive Kodaira dimension and X_{min} be its minimal model derived by $\pi : X \rightarrow X_{min}$ contracting all the (-1) -curves $E = \cup_i E_i$.

If $\mathrm{kod}(X) = 2$, X is a surface of general type. Let $\Phi : X_{min} \rightarrow X_{can}$ be holomorphic canonical map from X_{min} to its canonical model X_{can} with possible orbifold singularities. There exists a unique smooth orbifold Kähler-Einstein metric ω^\dagger on X_{can} . We define

$$\omega_{KE} = (\pi \circ \Phi)^* \omega^\dagger + \sqrt{-1} \partial \bar{\partial} \log |E|^2.$$

Choose a smooth positive $(1, 1)$ -form $\omega \in -c_1(X_{min})$ and a smooth volume form Ω with $\sqrt{-1}\partial\bar{\partial}\log\Omega = \omega$. Then $\Phi^*\omega^\dagger = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ for some $\varphi \in C^0(X_{min})$ satisfying the following Monge-Ampère equation

$$\frac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^2}{\Omega} = e^\varphi.$$

Furthermore, φ is smooth outside the exceptional locus of the pluricanonical system. Let $\Omega_{KE} = \pi^*(e^\varphi\Omega)$ the pullback of the unique holomorphic Kähler-Einstein volume from its canonical model. It is a continuous measure on X vanishing exactly on E of order one. Then $\omega_{KE} = Ric(\Omega_{KE})$ on X and $\omega_{KE}^2 = \Omega_{KE}$ on $X \setminus E$.

Proposition 4.1 *Let X be a Kähler surface of general type. Then*

1. $\omega_{KE} \in -c_1(X)$,
2. $Ric(\omega_{KE}) = -\omega_{KE}$ on $X \setminus E$,
3. $h_{KE} = \Omega_{KE}^{-1}$ is an analytic Zariski decomposition for K_X .

If $\text{kod}(X) = 1$, X is an elliptic surface. Let $\Phi : X_{min} \rightarrow X_{can}$ be the holomorphic pluricanonical fibration from X_{min} to its canonical model X_{can} . By Theorem 3.1 in Section 3.2, there exist a canonical metric ω^\dagger on X_{can} and a canonical measure Ω^\dagger on X_{min} such that $\Phi^*(\omega^\dagger) \in -c_1(X_{min})$ and $Ric(\Omega^\dagger) = \Phi^*(\omega^\dagger)$. Let $\Omega_{can} = \pi^*(\Omega^\dagger)$ and $\omega_{can} = Ric(\Omega_{can})$.

Proposition 4.2 *Let X be a Kähler surface of Kodaira dimension 1. Then*

1. $\omega_{can} \in -c_1(X)$,
2. $\omega_{can} = (\Phi \circ \pi)^*(\omega^\dagger)$ on $X \setminus E$.
3. $h_{can} = \Omega_{can}^{-1}$ is an analytic Zariski decomposition for K_X .

4.2 Ricci-flat metrics on Kähler manifolds of zero Kodaira dimension

There have been many interesting results on singular Ricci-flat metrics. In [EyGuZe1], singular Ricci-flat metrics are studied on normal Kähler spaces. In [To], singular Ricci-flat metrics are derived as the limit of smooth Ricci-flat Kähler metrics along certain degeneration of Kähler classes. In this section, we construct singular Ricci-flat metrics on algebraic manifolds of Kodaira dimension 0 as an immediate application of Theorem 2.4.

Let X be an n -dimensional algebraic manifold of Kodaira dimension 0. Suppose $L \rightarrow X$ is a holomorphic line bundle such that L is big and semi-ample. There exists a big smooth semi-positive $(1, 1)$ -form $\omega \in c_1(L)$. Let $\eta \in H^0(X, K_X^m)$ be the holomorphic m -tuple n -form for some $m \in N(K_X)$. Let

$$\Omega = \frac{[\omega]^n}{\int_X (\eta \otimes \bar{\eta})^{\frac{1}{m}}} (\eta \otimes \bar{\eta})^{\frac{1}{m}}$$

be a smooth (n, n) -form on X and so $\int_X \Omega = [\omega]^n$. Ω is independent of the choice of $m \in N(K_X)$ and η because the Kodaira dimension of X is 0. Ω is unique up to a scalar multiplication and it can be degenerate because K_X is not necessarily nef.

Consider the following degenerate Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = \Omega. \quad (4.1)$$

By Theorem 2.4, there exists a continuous solution φ to equation (4.1) unique up to a constant. Let $\omega_{CY} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$.

Proposition 4.3 *ω_{CY} is the unique closed semi-positive $(1, 1)$ -current in $c_1(L)$ with continuous local potential such that $\omega_{CY}^n = \Omega$ and therefore outside the base locus of the pluricanonical system*

$$\text{Ric}(\omega_{CY}) = 0.$$

Furthermore, ω_{CY} is smooth on a Zariski open set of X .

4.3 Kähler-Einstein metrics on algebraic manifolds of general type

In this section, we will prove Theorem B.1. Let X be an n -dimensional nonsingular algebraic variety of general type.

We choose a sequence of resolution of indeterminacies of the pluricanonical systems Φ_m

$$X \xleftarrow{\pi_{m_0}} X_{m_0} \xleftarrow{\pi_{m_0+1}} X_{m_0+1} \xleftarrow{\pi_{m_0+2}} \cdots \xleftarrow{\pi_m} X_m \xleftarrow{\pi_{m+1}} X_{m+1} \xleftarrow{\pi_{m+2}} \cdots \quad (4.2)$$

for m_0 sufficiently large, such that

1.

$$(\bar{\pi}_m)^*(m!K_X) = L_m + E_m,$$

where $\bar{\pi}_m = \pi_m \circ \pi_{m-1} \circ \cdots \circ \pi_{m_0}$.

2.

$$E_m = \sum_j c_{m,j} E_{m,j}$$

is the fixed part of $|\bar{\pi}_m^*(m!K_X)|$ with each $E_{m,j}$ being a divisor with simple normal crossings.

3. L_m is a globally generated line bundle on X_m .

Let $\{\sigma_{m,j}\}_{j=0}^{d_m}$ be a basis of $H^0(X, m!K_X)$ and $\{\zeta_{m,j}\}_{j=0}^{d_m}$ be a basis of $H^0(X_m, L_m)$ such that

$$\pi_m^* \sigma_{m,j} = \zeta_{m,j} E_m.$$

We can consider $|\sigma_{m,j}|^{\frac{2}{m!}}$ as a smooth (n, n) -form on X as $\sigma_{m,j} \in m!K_X$. Let $\Omega_m = \left(\sum_{j=0}^{d_m} |\sigma_{m,j}|^2\right)^{\frac{1}{m!}}$ and then $\pi_k^* \Omega_m$ is a smooth and possibly degenerate volume form on X_k . For simplicity we also use Ω_m for $(\pi_k)^*(\Omega_m)$ for all $k \geq m_0$.

The following lemma is obviously by the construction of Ω_m .

Lemma 4.1 *On X and so on X_k for $k \geq m_0$,*

$$\frac{\Omega_m}{\Omega_{m+1}} < \infty. \quad (4.3)$$

Define a smooth closed semi-positive $(1, 1)$ -form ω_m on X_m by

$$\omega_m = \frac{1}{m!} \sqrt{-1} \partial \bar{\partial} \log \left(\sum_{j=0}^{d_m} |\zeta_j|^2 \right).$$

Obviously $\sqrt{-1} \partial \bar{\partial} \log \left(\sum_{j=0}^{d_m} |\zeta_j|^2 \right)$ is the pullback of the Fubini-Study metric on \mathbf{CP}^{d_m} from the linear system $|L_m|$.

Theorem 4.1 *Let R_m be the exceptional locus of the linear systems associated to L_m . There exists a unique solution $\varphi_m \in C^0(X_m) \cap C^\infty(X_m \setminus (\cup_j E_{m,j} \cup R_m))$ to the following Monge-Ampère equation*

$$(\omega_m + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^\varphi \Omega_m. \quad (4.4)$$

Therefore $\omega_{KE,m} = \omega_m + \sqrt{-1} \partial \bar{\partial} \varphi_m$ is a Kähler-Einstein current on X_m satisfying

1. $\omega_{KE,m}$ is a positive current on X_m and strictly positive on $X_m \setminus (\cup_j E_{m,j} \cup R_m)$,
2. $\text{Ric}(\omega_{KE,m}) = -\omega_{KE,m}$ on $X_m \setminus (\cup_j E_{m,j})$.

Proof Let $F_m = \frac{\Omega_m}{(\omega_m)^n}$. F_m has at worst pole singularities on X_m and

$$\int_{X_m} (F_m)^{1+\epsilon} (\omega_m)^n = \int_{X_m} (F_m)^\epsilon \Omega_m < \infty.$$

By Theorem 2.4, there exists a unique $\varphi_m \in PSH(X_m, \omega_m) \cap C^0(X_m)$ solving the equation (4.4).

Also L_m is an semi-ample line bundle and furthermore it is big. Then by Kodaira's Lemma, there exists a divisor $[F_m]$ such that

$$[L_m] - \epsilon[F_m] = \pi_m^*[K_X] - \sum_j \frac{c_{m,j}}{m} [E_{m,j}] - \epsilon[F_m]$$

is ample for sufficiently small $\epsilon > 0$. By a similar argument in the proof of Theorem 3.2, we can show that $\varphi_m \in C^\infty(X_m \setminus (\cup_j E_{m,j} \cup R_m))$. □

Corollary 4.1 *$e^{\varphi_m} \Omega_m$ descends from X_m to a continuous measure on X and $h_m = e^{-\varphi_m} \Omega_m^{-1}$ is a singular hermitian metric on X with its curvature $\Theta_{h_m} \geq 0$. Furthermore, on $X \setminus Bs(|m_0 K_X|)$,*

$$(\sqrt{-1} \partial \bar{\partial} \log \Omega_m + \sqrt{-1} \partial \bar{\partial} \varphi_m)^n = e^{\varphi_m} \Omega_m. \quad (4.5)$$

Proof $e^{\varphi_m} \Omega_m$ can be pulled back as a continuous volume form on $X \setminus Bs(m!K_X)$. On the other hand, φ_m is uniformly bounded in $L^\infty(X)$, also Ω_m is smooth on X and vanishes exactly on $Bs(|m!K_X|)$. Therefore $e^{\varphi_m} \Omega_m$ is continuous on X .

$Bs(|m!K_X|)$ is a complete closed pluripolar set on X using $H^0(X, m!K_m)$. Since $\log \Omega_m + \varphi_m$ is uniformly bounded above and $\sqrt{-1} \partial \bar{\partial} \log \Omega_m + \sqrt{-1} \partial \bar{\partial} \varphi_m$ is a positive closed $(1, 1)$ current on $X \setminus Bs(|m_0 K_X|)$, $\sqrt{-1} \partial \bar{\partial} \log \Omega_m + \sqrt{-1} \partial \bar{\partial} \varphi_m$ extends to a positive closed current on X using local argument.

Equation (4.5) is then derived directly from Equation (4.4). □

Although Θ_{h_m} is a singular Kähler-Einstein metric on X , but without assuming finite generation of canonical rings, (K_X, h_m) is not necessarily an analytic Zariski decomposition. One has to let m tend to infinity in order for $e^{\varphi_m} \Omega_m$ to have the least vanishing order.

Let D be an ample divisor on X such that $K_X + D$ is ample. Then there exists a hermitian metric h_D on the line bundle induced by $[D]$ such that $\omega_0 - \sqrt{-1} \partial \bar{\partial} \log h_D > 0$. We can also assume that D contains the base locus of all $|m!K_X|$ for $m \geq m_0$.

Lemma 4.2 *Let Ω_0 be a smooth and nowhere vanishing volume form on X . Then there exists a constant $C > 0$ such that for each $m \geq m_0$,*

$$e^{\varphi_m} \Omega_m \leq C \Omega_0.$$

Proof Let $\chi_0 = \sqrt{-1} \partial \bar{\partial} \log \Omega_0$. Let D_m be a divisor on X such that on $X \setminus D_m$, φ_m is smooth and Ω_m is strictly positive. Let D be an ample divisor on X . Let S_{D_m} be a defining function and h_{D_m} be a smooth hermitian metric on the line bundle associated to $[D_m]$. Let S_D be a defining function and h_D be a smooth positively curved hermitian metric on the line bundle associated to $[D]$.

Let $\theta_{D_m} = -\sqrt{-1} \partial \bar{\partial} \log h_{D_m}$ and $\theta_D = -\sqrt{-1} \partial \bar{\partial} \log h_D > 0$. We also define

$$\psi_{m,\epsilon} = \varphi_m + \log \frac{\Omega_m}{\Omega_0} + \epsilon^2 \log |S_{D_m}|_{h_m}^2 + \epsilon \log |S_D|_{h_D}^2.$$

For simplicity, we use χ_0 and Ω_0 for $(\bar{\pi}_m)^* \chi_0$ and $(\bar{\pi}_m)^* \Omega_0$. Notice that χ_0 is not necessarily positive and Ω_0 might vanish somewhere on X_m . Then outside D_m and D , $\psi_{m,\epsilon}$ satisfies the following equation

$$\frac{(\chi_0 + \epsilon^2 \theta_{D_m} + \epsilon \theta_D + \sqrt{-1} \partial \bar{\partial} \log \psi_{m,\epsilon})^n}{\Omega_0} = |S_{D_m}|_{h_{D_m}}^{-2\epsilon^2} |S_D|_{h_D}^{-2\epsilon} e^{\psi_{m,\epsilon}}.$$

It is easy to see that $\psi_{m,\epsilon}$ tends to $-\infty$ near D_m and D , and for $\epsilon > 0$ sufficiently small, $\theta_D + \epsilon \theta_{D_m} > 0$. By the maximum principle

$$e^{\psi_{m,\epsilon}} \leq \max_X \left(|S_{D_m}|_{h_{D_m}}^{2\epsilon^2} |S_D|_{h_D}^{2\epsilon} \frac{(\chi_0 + \epsilon^2 \theta_{D_m} + \epsilon \theta_D)^n}{\Omega_0} \right)$$

and

$$e^{\varphi_m} \frac{\Omega_m}{\Omega_0} \leq |S_{D_m}|_{h_{D_m}}^{-2\epsilon^2} |S_D|_{h_D}^{-2\epsilon} \max_X \left(|S_{D_m}|_{h_{D_m}}^{2\epsilon^2} |S_D|_{h_D}^{2\epsilon} \frac{(\chi_0 + \epsilon^2 \theta_{D_m} + \epsilon \theta_D)^n}{\Omega_0} \right).$$

Now we let ϵ tend to 0, and

$$e^{\varphi_m} \frac{\Omega_m}{\Omega_0} \leq \max_X \left(\frac{(\chi_0)^n}{\Omega_0} \right).$$

□

Proposition 4.4 $e^{\varphi_m} \Omega_m$ is increasing, that is, on X

$$e^{\varphi_m} \Omega_m \leq e^{\varphi_{m+1}} \Omega_{m+1}. \quad (4.6)$$

Proof We shall compare $e^{\varphi_m} \Omega_m$ and $e^{\varphi_{m+1}} \Omega_{m+1}$ on X_{m+1} . Let

$$U_{(m+1)} = \left\{ s \in X_{m+1} \mid \frac{\Omega_{m+1}}{\Omega_m} < \infty \right\}.$$

φ_m and φ_{m+1} are the solutions of

$$(\omega_m + \sqrt{-1} \partial \bar{\partial} \varphi_m)^n = e^{\varphi_{m+1}} \Omega_m.$$

and

$$(\omega_{m+1} + \sqrt{-1} \partial \bar{\partial} \varphi_{m+1})^n = e^{\varphi_{m+1}} \Omega_{m+1}.$$

Define

$$\psi = \varphi_{m+1} - \varphi_m + \log \frac{\Omega_m}{\Omega_{m+1}}$$

and

$$V = \{z \in X_{m+1} \mid \psi < 0\}.$$

It is easy to see that

$$V \subset U_{m+1}$$

since both φ_{m+1} and φ_m are in $L^\infty(X_{m+1})$.

On U_{m+1} , $\log \left(\frac{(E_{m+1})^{m!}}{((\pi_{m+1})^* E_m)^{(m+1)!}} \right)$ is holomorphic and so

$$\begin{aligned} & \omega_{m+1} + \sqrt{-1} \partial \bar{\partial} \varphi_{m+1} \\ = & \omega_m + \sqrt{-1} \partial \bar{\partial} \varphi_m + \sqrt{-1} \partial \bar{\partial} \log(\varphi_{m+1} - \varphi_m) + \sqrt{-1} \partial \bar{\partial} \log \left(\frac{\left(\sum_{j=0}^{d_{m+1}} |\zeta_{m+1,j}|^2 \right)^{\frac{1}{(m+1)!}}}{\left(\sum_{j=0}^{d_m} |(\pi_{m+1})^* \zeta_{m,j}|^2 \right)^{\frac{1}{m!}}} \right) \\ = & \omega_m + \sqrt{-1} \partial \bar{\partial} \varphi_m + \sqrt{-1} \partial \bar{\partial} \psi + \sqrt{-1} \partial \bar{\partial} \log \left(\frac{|E_{m+1}|^{\frac{2}{(m+1)!}}}{|(\pi_{m+1})^* E_m|^{\frac{2}{m!}}} \right) \\ = & \omega_m + \sqrt{-1} \partial \bar{\partial} \varphi_m + \sqrt{-1} \partial \bar{\partial} \psi. \end{aligned}$$

Hence ψ satisfies the following equation on U_{m+1}

$$\frac{(\omega_m + \sqrt{-1}\partial\bar{\partial}\varphi_m + \sqrt{-1}\partial\bar{\partial}\psi)^n}{(\omega_m + \sqrt{-1}\partial\bar{\partial}\varphi_m)^n} = e^\psi. \quad (4.7)$$

By the comparison test,

$$\begin{aligned} & \int_V (\omega_m + \sqrt{-1}\partial\bar{\partial}\varphi_m)^n \\ & \leq \int_V (\omega_m + \sqrt{-1}\partial\bar{\partial}\varphi_m + \sqrt{-1}\partial\bar{\partial}\psi)^n \\ & = \int_V e^\psi (\omega_m + \sqrt{-1}\partial\bar{\partial}\varphi_m)^n. \end{aligned}$$

Therefore $\Psi = 0$ on V due to the fact that it is continuous on V , and so on X_{m+1}

$$\psi \geq 0.$$

This completes the proof. □

Proposition 4.5 *There exists a measure Ω_{KE} on X such that*

1. $\Omega_{KE} = \lim_{m \rightarrow \infty} e^{\varphi_m} \Omega_m$.
2. (K_X, Ω_{KE}^{-1}) is an analytic Zariski decomposition. Furthermore, on X

$$\frac{\Omega_{KE}}{\Omega_0} < \infty \quad \text{and} \quad \frac{\Psi_{X,\epsilon}}{\Omega_{KE}} < \infty.$$

3. $\omega_{KE} = \sqrt{-1}\partial\bar{\partial} \log \Omega_{KE} \in -c_1(X)$ is a closed positive Kähler-Einstein current and on $X \setminus Bs(X, K_X)$,

$$(\omega_{KE})^n = \Omega_{KE} \quad \text{and so} \quad \text{Ric}(\omega_{KE}) = -\omega_{KE}.$$

Proof By Lemma 4.2 and Proposition 4.4, we can define Ω_{KE}

$$\Omega_{KE} = \lim_{m \rightarrow \infty} e^{\varphi_m} \Omega_m$$

and $\frac{\Omega_m}{\Omega_0}$ is uniformly bounded from above. Since $\varphi_m + \log \frac{\Omega_m}{\Omega_0} \in PSH(X, \chi_0)$ and $\{\varphi_m + \log \frac{\Omega_m}{\Omega_0}\}_{m=m_0}^\infty$ is convergent in $L^1(X)$. Also $PSH(X, \chi_0) \cap L^1(X)$ is closed in $L^1(X)$. Therefore

$$\lim_{m \rightarrow \infty} \left(\varphi_m + \log \frac{\Omega_m}{\Omega_0} \right) = \log \frac{\Omega_{KE}}{\Omega_0}$$

in $PSH(X, \chi_0) \cap L^1(X)$ and $\log \frac{\Omega_{KE}}{\Omega_0} \in PSH(X, \chi_0) \cap L^1(X)$.

Let $h_{KE} = \Omega_{KE}^{-1}$ be the hermitian metric on K_X . By the construction of Ω_m ,

$$|\sigma|_{h_{KE}^m}^2 < \infty$$

for any section $\sigma \in H^0(X, mK_X)$. Therefore $\frac{\Psi_{X,\epsilon}}{\Omega_{KE}} < \infty$ and so

$$H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h_{KE}^m)) \rightarrow H^0(X, \mathcal{O}_X(mK_X))$$

is an isomorphism and (K_X, h_{KE}) is an analytic Zariski decomposition.

Since $\varphi_m + \log \frac{\Omega_m}{\Omega_0}$ converges uniformly on any compact set of $X \setminus Bs(X, K_X)$ to $\log \frac{\Omega_{KE}}{\Omega_0}$, we have on $X \setminus Bs(X, K_X)$,

$$(\sqrt{-1} \partial \bar{\partial} \log \Omega_{KE})^n = \lim_{m \rightarrow \infty} (\sqrt{-1} \partial \bar{\partial} \log \Omega_m + \sqrt{-1} \partial \bar{\partial} \varphi_m)^n = \lim_{m \rightarrow \infty} e^{\varphi_m} \Omega_m = \Omega_{KE}.$$

This concludes the proof of the proposition as well as Theorem B.1. \square

Remark 4.1 The existence of such a canonical metric does not depend on the finite generation of the canonical ring of X . The regularity and uniqueness of such Kähler-Einstein metrics will be investigated in our future study.

4.4 Algebraic manifolds of positive Kodaira dimension

Let X be an n -dimensional nonsingular algebraic variety of Kodaira dimension κ , where $0 < \kappa < n$. Let $\Phi^\dagger : X^\dagger \rightarrow Y^\dagger$ be the Iitaka fibration of X unique up to birational equivalence. Let Φ_m be the pluricanonical map associated to the linear system $|mK_X|$. Then for m sufficiently large there exists a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\pi^\dagger} & X^\dagger \\ \Phi_m \downarrow \vdots & & \downarrow \Phi^\dagger \\ Y_m & \xleftarrow{\mu_m} & Y^\dagger \end{array} \quad (4.8)$$

as in Section 2.2. A very general fibre of Φ^\dagger has Kodaira dimension zero.

We will generalize the notion of the Weil-Petersson metric on a special local deformation space of Kähler manifolds of zero Kodaira dimension.

Definition 4.1 Let $f : \mathcal{X} \rightarrow B$ be a holomorphic nonsingular fibration over a ball $B \in \mathbb{C}^\kappa$ such that for any $t \in B$, $\mathcal{X}_t = f^{-1}(t)$ is a nonsingular fibre of dimension $n - \kappa$. Let $t = (t_1, \dots, t_\kappa)$ be the holomorphic coordinates of B , where $\kappa = \dim \mathcal{M}$. Then each $\frac{\partial}{\partial t_i}$ corresponds to an element $\iota(\frac{\partial}{\partial t_i}) \in H^1(\mathcal{X}_t, T_{\mathcal{X}_t})$ through the Kodaira-Spencer map ι . We assume that there exists a nontrivial holomorphic $(n - \kappa, 0)$ -form Ψ on \mathcal{X} such that its restriction on each fibre \mathcal{X}_t is also a nontrivial holomorphic $(n - \kappa, 0)$ -form on \mathcal{X} . Then the Weil-Petersson metric is defined by the L^2 -inner product

$$\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \right)_{\omega_{WP}} = \frac{\int_{\mathcal{X}_t} \Psi \lrcorner \iota(\frac{\partial}{\partial t_i}) \wedge \overline{\Psi \lrcorner \iota(\frac{\partial}{\partial t_j})}}{\int_{\mathcal{X}_t} \Psi \wedge \overline{\Psi}}, \quad (4.9)$$

where $\Psi \lrcorner \iota(\frac{\partial}{\partial t_i})$ is the contraction of Ψ and $\iota(\frac{\partial}{\partial t_i})$.

The metric defined above is a pseudometric and the associated closed $(1, 1)$ -form ω_{WP} is only semi-positive in general. In Definition 4.1, ω_{WP} depends on the choice of the holomorphic $(n - \kappa, 0)$ -form Ψ , however, it is uniquely determined for the Iitaka fibration $\Phi^\dagger : X^\dagger \rightarrow Y^\dagger$ by the following lemma.

Lemma 4.3 *The Weil-Petersson metric is well-defined on a Zariski open set of Y^\dagger and it is unique.*

Proof Let $B \subset Y^\dagger$ be a nonsingular open neighborhood such that each fibre over B is nonsingular. Let $\mathcal{X} = (\Phi^\dagger)^{-1}(B) \subset X^\dagger$. Without loss of generality, we assume $H^0(X^\dagger, K_{X^\dagger}) \neq \phi$, otherwise we can replace K_X by a sufficiently large power of K_X , and so $H^0(B, (\Phi^\dagger)_* \Omega_{\mathcal{X}/B}^{n-\kappa}) \neq \phi$. Then the assumption in Definition 4.1 can be satisfied. Let Ψ_1 and Ψ_2 be two holomorphic $(n - \kappa)$ -forms over B in Definition 4.1. Since a very general fibre has Kodaira dimension zero, for a general point $t \in B$,

$$\frac{\Psi_1}{\Psi_2} \Big|_{\mathcal{X}_t} = \text{constant}$$

and so $\frac{\Psi_1}{\Psi_2} \Big|_{\mathcal{X}_t}$ is constant on each point $t \in B$ since $\frac{\Psi_1}{\Psi_2} \Big|_{\mathcal{X}_t}$ is smooth on \mathcal{X} .

One can also represent ω_{WP} as the curvature form of the first Hodge bundle $f_* \Omega_{\mathcal{X}/B}^{n-\kappa}$ with the same assumption as in Definition 4.1. Let Ψ be a nonzero local holomorphic section of $f_* \Omega_{\mathcal{X}/B}^{n-\kappa}$ and one can define the hermitian metric h_{WP} on $f_* \Omega_{\mathcal{X}/B}^{n-\kappa}$ by

$$|\Psi_t|_{h_{WP}}^2 = \int_{\mathcal{X}_t} \Psi_t \wedge \overline{\Psi}_t. \quad (4.10)$$

Then the Weil-Petersson metric is given by

$$\omega_{WP} = \text{Ric}(h_{WP}). \quad (4.11)$$

Therefore the Weil-Petersson metric is unique on B . □

We choose a sequence of resolution of indeterminacies of the pluricanonical systems $\Phi_{m!} = \Phi_{|m!K_X|}$

$$X \xleftarrow{\pi_{m_0}} X_{m_0} \xleftarrow{\pi_{m_0+1}} X_{m_0+1} \xleftarrow{\pi_{m_0+2}} \cdots \xleftarrow{\pi_m} X_m \xleftarrow{\pi_{m+1}} X_{m+1} \xleftarrow{\pi_{m+2}} \cdots \quad (4.12)$$

for m_0 sufficiently large, such that

1.

$$(\bar{\pi}_m)^*(m!K_X) = L_m + E_m,$$

where $\bar{\pi}_m = \pi_m \circ \pi_{m-1} \circ \cdots \circ \pi_{m_0}$.

2.

$$E_m = \sum_j c_{m,j} E_{m,j}$$

is the fixed part of $|(\bar{\pi}_m)^*(m!K_X)|$ with each $E_{m,j}$ being a divisor with simple normal crossings.

3. L_m is a globally generated line bundle on X_m .

Let Y_m be the variety determined by the pluricanonical system $|m!K_X|$ and $\Psi_m = \Psi_{|L_m|}$ be the rational map associated to the linear system $|L_m|$. Then we have the following diagram

$$\begin{array}{ccc} X & \xleftarrow{\bar{\pi}_m} & X_m \\ & \searrow \Phi_m & \swarrow \Psi_m \\ & & Y_m \end{array} \quad (4.13)$$

There exists a commutative diagram by choosing m_0 sufficiently large

$$\begin{array}{ccccccc} X & \xleftarrow{\pi_{m_0}} & X_{m_0} & \xleftarrow{\pi_{m_0+1}} & X_{m_0+1} & \xleftarrow{\pi_{m_0+2}} & \cdots \xleftarrow{\pi_m} X_m \xleftarrow{\pi_{m+1}} X_{m+1} \xleftarrow{\pi_{m+2}} \cdots \\ & & \downarrow \Psi_{m_0} & & \downarrow \Psi_{m_0+1} & & \downarrow \Psi_m \\ & & Y_{m_0} & \xleftarrow{\mu_{m_0+1}} & Y_{m_0+1} & \xleftarrow{\mu_{m_0+2}} & \cdots \xleftarrow{\mu_m} Y_m \xleftarrow{\mu_{m+1}} Y_{m+1} \xleftarrow{\mu_{m+2}} \cdots \end{array} \quad (4.14)$$

of rational maps and holomorphic maps where the horizontal maps are birational and μ_m is given by the projection from $|m!K_X|$ to $|(m-1)!K_X|$ as a subspace of $|m!K_X|$.

Let $\{\sigma_{m,j}\}_{j=0}^{d_m}$ be a basis of $H^0(X, m!K_X)$ and $\{\zeta_{m,j}\}_{j=0}^{d_m}$ be a basis of $H^0(X, L_m)$ such that

$$(\bar{\pi}_m)^* \sigma_{m,j} = \zeta_{m,j} E_m \in H^0(X_m, (\bar{\pi}_m)^*(m!K_X)).$$

Then

$$\frac{\left(\sum_{j=0}^{d_{m+1}} \sigma_{m+1,j} \otimes \overline{\sigma_{m+1,j}} \right)^{\frac{1}{(m+1)!}}}{\left(\sum_{j=0}^{d_m} \sigma_{m,j} \otimes \overline{\sigma_{m,j}} \right)^{\frac{1}{m!}}} < \infty,$$

where $1 + d_m = \dim H^0(X, m!K_X)$.

Let $\Omega_m = \left(\sum_{j=0}^{d_m} \sigma_{m,j} \otimes \overline{\sigma_{m,j}} \right)^{\frac{1}{m!}}$ and $\Omega_{(m)} = (\Psi_m)_* \Omega_m$. Let

$$\omega_m = \frac{1}{m} \sqrt{-1} \partial \bar{\partial} \log \left(\sum_{j=0}^{d_m} |\zeta_{m,j}|^2 \right)$$

be the normalized Fubini-Study metric on Y_m . Then the same argument in Proposition 3.2 gives the following Lemma.

Lemma 4.4 *There exists $p = p(m) > 1$ such that*

$$F_m = \frac{\Omega_{(m)}}{\omega_m^\kappa} \in L^p(Y_m, \omega_m^\kappa). \quad (4.15)$$

The following proposition is immediate as in Section 3.2.

Proposition 4.6 *There exists a unique solution $\varphi_m \in PSH(Y_m, \omega_m) \cap C^0(Y_m)$ to the following Monge-Ampère equation*

$$(\omega_m + \sqrt{-1}\partial\bar{\partial}\varphi_m)^\kappa = e^{\varphi_m}\Omega_{(m)}. \quad (4.16)$$

Furthermore, φ_m is smooth on a Zariski open set of Y_m .

For simplicity, we abuse the notations and use Ω_m for $(\bar{\pi}_k)^*\Omega_m$ for $k \geq m_0$ without causing confusion.

Lemma 4.5 *There exists a Zariski open set U of Y_{m+1} such that $\frac{\Omega_m}{\Omega_{m+1}}$ is constant along each fibre of Ψ_{m+1} over U .*

Proof Let $F = \frac{\Omega_m}{\Omega_{m+1}}$ and it is easy to see that F is smooth. We consider the following diagram

$$\begin{array}{ccc} X_{m+1} & \xleftarrow{f} \cdots \cdots & X^\dagger \\ \Psi_{m+1} \downarrow & & \downarrow \Phi^\dagger \\ Y_{m+1} & \xleftarrow{g} \cdots \cdots & Y^\dagger \end{array}$$

where $\Phi^\dagger : X^\dagger \rightarrow Y^\dagger$ is an Iitaka fibration of X .

A very general fiber of Φ^\dagger is nonsingular of Kodaira dimension 0. Let $F_{s_0} = (\Phi^\dagger)^{-1}(s_0)$ be a very general fibre. Consider $B = \{s \in Y^\dagger \mid |s - s_0| < \delta\}$ such that for any $s \in B$, $F_s = (\Phi^\dagger)^{-1}(s)$ is non-singular. Let η be a nowhere-vanishing holomorphic κ -form on B . Then

$$\left. \frac{(\pi^\dagger)^*\sigma_{m,j}}{\eta^m} \right|_{s_0} \in H^0(F_{s_0}, m!K_{F_{s_0}})$$

and

$$\left. \frac{(\pi^\dagger)^*\sigma_{m+1,j}}{\eta^{m+1}} \right|_{s_0} \in H^0(F_{s_0}, (m+1)!K_{F_{s_0}}).$$

Since $\dim PH^0(F_{s_0}, kK_{F_{s_0}}) = 0$ for any $k \geq 1$ and $\frac{\Omega_m}{\Omega_{m+1}} < \infty$, $\frac{\Omega_m}{\Omega_{m+1}}$ must be constant on each F_{s_0} .

Therefore f^*F is constant on a very general fibre of Φ^\dagger . By Hartog's theorem, f^*F is smooth on X^\dagger and so f^*F is the pullback of a function on a Zariski open set of Y^\dagger . By the commutative diagram, on a Zariski open set of X_{m+1} , F is the pullback of function on Y_{m+1} and so F has to be constant on a very general fibre of Ψ_{m+1} . □

Hence $\frac{\Omega_m}{\Omega_{m+1}}$ descends to a function on Y_{m+1} . By the commutative diagram, the following lemma is immediate.

Lemma 4.6 *For each $m \geq m_0$,*

$$(\psi_{m+1})_* \Omega_m = (\mu_{m+1})^* \Omega_{(m)}.$$

The following corollary is immediate by the diagram 4.14, Lemma 4.5 and 4.6.

Corollary 4.2 *For each $m \geq m_0$, on X_{m+1}*

$$\frac{\Omega_m}{\Omega_{m+1}} = \frac{(\mu_{m+1})^* \Omega_{(m)}}{\Omega_{(m+1)}}.$$

Lemma 4.7 *Let $U_{m+1} = \{s \in Y_{m+1} \mid \frac{\Omega_{m+1}}{\Omega_m} < \infty\}$. Then μ_{m+1} is holomorphic on U_{m+1} .*

Proof On $(\Psi_{m+1})^{-1}(U_{m+1})$,

$$0 < \frac{\left(\sum_j |\sigma_{m,j}|^2\right)^{\frac{1}{m}}}{\left(\sum_j |\sigma_{m+1,j}|^2\right)^{\frac{1}{m+1}}} < \infty.$$

Both L_{m+1} and $(\pi_m)^* L_m$ are globally generated. Therefore the base locus of

$$\left\{ \frac{((\bar{\pi}_{m+1})^* \sigma_{m,j})^{m+1}}{E_{m+1}} \right\}_{j=0}^{d_m}$$

is outside U_{m+1} and so μ_{m+1} is well defined on U_{m+1} . □

Proposition 4.7 *For any $m \geq m_0$, the measure $e^{\varphi_{(m)}} \Omega_{(m)}$ is increasing, that is, on Y_{m+1}*

$$(\mu_{m+1})^* (e^{\varphi_{(m)}} \Omega_{(m)}) \leq e^{\varphi_{(m+1)}} \Omega_{(m+1)}. \quad (4.17)$$

Proof By resolution of singularities, we can assume $Y_{(m+1)}$ is non-singular by replacing Y_{m+1} by its non-singular model. Let $\omega'_m = (\mu_{m+1})^* \omega_m$, $\varphi'_m = (\mu_{m+1})^* \varphi_m$ and $\Omega'_{(m)} = (\mu_{m+1})^* \Omega_{(m)}$. So φ'_m satisfies the following Monge-Ampère equation

$$(\omega'_m + \sqrt{-1} \partial \bar{\partial} \varphi'_m)^\kappa = e^{\varphi'_m} \Omega'_{(m)}$$

on U_{m+1} , where the ω'_m is a smooth and positive, and so the Monge-Ampère mass $(\omega'_m + \sqrt{-1} \partial \bar{\partial} \varphi'_m)^\kappa$ is well-defined. Also φ_{m+1} is the solution of

$$(\omega_{m+1} + \sqrt{-1} \partial \bar{\partial} \varphi_{m+1})^\kappa = e^{\varphi_{m+1}} \Omega_{(m+1)}.$$

Define

$$\psi = \varphi_{m+1} - \varphi'_m + \log \frac{\Omega_{(m+1)}}{\Omega'_{(m)}}$$

and $V = \{z \in Y_m \mid \psi \leq 0\}$. It is easy to see that

$$V \subset U_{m+1}.$$

since both φ_{m+1} and φ'_m are in $L^\infty(Y_{m+1})$. In particular, $\varphi'_m \in C^0(U_{m+1})$.

On U_{m+1} , $\log \left(\frac{|E_{m+1}|^{2m}}{|(\pi_m)^* E_m|^{2(m+1)}} \right)$ is smooth and so

$$\begin{aligned} & \omega_{m+1} + \sqrt{-1} \partial \bar{\partial} \varphi_{m+1} \\ = & \omega'_m + \sqrt{-1} \partial \bar{\partial} \varphi'_m + \sqrt{-1} \partial \bar{\partial} \log(\varphi_{m+1} - \varphi'_m) + \sqrt{-1} \partial \bar{\partial} \log \left(\frac{\left(\sum_j |\zeta_{m+1,j}|^2 \right)^{\frac{1}{m+1}}}{\left(\sum_j |(\pi_m)^* \zeta_{m,j}|^2 \right)^{\frac{1}{m}}} \right) \\ = & \omega'_m + \sqrt{-1} \partial \bar{\partial} \varphi'_m + \sqrt{-1} \partial \bar{\partial} \psi - \sqrt{-1} \partial \bar{\partial} \log \left(\frac{|E_{m+1}|^{\frac{2}{m+1}}}{|(\pi_m)^* E_m|^{\frac{2}{m}}} \right) \\ = & \omega'_m + \sqrt{-1} \partial \bar{\partial} \varphi'_m + \sqrt{-1} \partial \bar{\partial} \psi. \end{aligned}$$

Hence ψ satisfies the following equation on U_{m+1}

$$\frac{(\omega'_m + \sqrt{-1} \partial \bar{\partial} \varphi'_m + \sqrt{-1} \partial \bar{\partial} \psi)^\kappa}{(\omega'_m + \sqrt{-1} \partial \bar{\partial} \varphi'_m)^\kappa} = e^\psi. \quad (4.18)$$

By the comparison test, we have on $V = \{z \in Y_{m+1} \mid \psi(z) < 0\} \subset U_{m+1}$

$$\begin{aligned} & \int_V (\omega'_m + \sqrt{-1} \partial \bar{\partial} \varphi'_m)^\kappa \\ \leq & \int_V (\omega'_m + \sqrt{-1} \partial \bar{\partial} \varphi'_m + \sqrt{-1} \partial \bar{\partial} \psi)^\kappa \\ = & \int_V e^\psi (\omega'_m + \sqrt{-1} \partial \bar{\partial} \varphi'_m)^\kappa. \end{aligned}$$

Therefore $\psi = 0$ on V due to the fact that it is continuous on V , and so on Y_{m+1}

$$\psi \geq 0.$$

This completes the proof. □

For simplicity we again use φ_m for $(\Psi_m)^* \varphi_m$. We can consider φ_m as a function on X , X_m or X^\dagger since they are all birationally equivalent.

Corollary 4.3 *For any $m \geq m_0$, the measure $e^{\varphi_m} \Omega_m$ is increasing, that is,*

$$e^{\varphi_m} \Omega_m \leq e^{\varphi_{m+1}} \Omega_{m+1}. \quad (4.19)$$

Lemma 4.8 $e^{\varphi_m} \Omega_m$ is continuous on X and let $h_m = e^{-\varphi_m} \Omega_m^{-1}$ be the hermitian metric on K_X . Then

$$\Theta_{h_m} \geq 0.$$

Proof Obviously, $e^{\varphi_m} \Omega_m$ is continuous on $X \setminus Bs(|m!K_X|)$ since the inverse of $\bar{\pi}_m$ is isomorphic from $X_m \setminus (\bar{\pi}_m)^{-1}(Bs(|m!K_X|))$ to $X \setminus (Bs(|m!K_X|))$. On the other hand, $\varphi_m \in L^\infty(X)$ and Ω_m vanishes exactly on $Bs(|m!K_X|)$. Hence $e^{\varphi_m} \Omega_m$ is continuous.

$\sqrt{-1} \partial \bar{\partial} \log \Omega_m + \sqrt{-1} \partial \bar{\partial} \varphi_m \geq 0$ on $X \setminus Bs(|m!K_X|)$ and $Bs(|m!K_X|)$ is a closed complete pluripolar set of X . By the fact that $e^{\varphi_m} \Omega_m$ is bounded above on X ,

$$\Theta_{h_m} = \sqrt{-1} \partial \bar{\partial} \log \Omega_m + \sqrt{-1} \partial \bar{\partial} \varphi_m$$

extends to a closed positive current on X by local argument. This completes the proof of the lemma. \square

Lemma 4.9 Let Ω be a smooth volume form on X^\dagger . Then there exist a divisor D of X^\dagger and a constant $C > 0$ such that

$$e^{\varphi_m} \Omega_m \leq |S_D|_{h_D}^{-2} \Omega, \quad (4.20)$$

where S_D is a defining section of $[D]$ and h_D is a fixed smooth hermitian metric of the line bundle associated to $[D]$.

Proof We consider again the following commutative diagram

$$\begin{array}{ccc} X_m & \xleftarrow{f_m} & X^\dagger \\ \Psi_m \downarrow & & \downarrow \Phi^\dagger \\ Y_m & \xleftarrow{g_m} & Y^\dagger \end{array}$$

Let ω be a Kähler metric and Ω_0 be a smooth and nowhere vanishing volume form on X^\dagger . From the commutative diagram, on a Zariski open set of X^\dagger we have

$$(\sqrt{-1} \partial \bar{\partial} \log \Omega_m + \sqrt{-1} \partial \bar{\partial} \varphi_m)^\kappa \wedge \omega^{n-\kappa} = e^{\varphi_m} (\Phi^\dagger)^* (\Phi^\dagger)_* \Omega_m \wedge \omega^{n-\kappa}.$$

Let $\psi_m = \varphi_m + \log \frac{\Omega_m}{\Omega}$, $\chi_0 = \sqrt{-1} \partial \bar{\partial} \log \Omega_0$.

ψ_m satisfies the following Monge-Ampère equation on $X^\dagger \setminus (\pi^\dagger)^{-1} Bs(|m_0!K_X|)$,

$$(\chi_0 + \sqrt{-1} \partial \bar{\partial} \psi_m)^\kappa \wedge \omega^{n-\kappa} = F e^{\psi_m} \Omega_0,$$

where

$$F = \frac{(\Phi^\dagger)^* (\Phi^\dagger)_* \Omega_m \wedge \omega^{n-\kappa}}{\Omega_m} = \frac{(\Phi^\dagger)^* (\Phi^\dagger)_* \Omega_{m_0} \wedge \omega^{n-\kappa}}{\Omega_{m_0}}.$$

Let D_1 be a divisor of X^\dagger containing $(\pi^\dagger)^{-1} Bs(|m_0!K_X|)$ such that the defining section S_1 of $[D_1]$ satisfies

$$|S_1|_{h_1}^2 \frac{\omega^n}{F\Omega_0} < \infty,$$

where h_1 is a fixed smooth hermitian metric of the line bundle associated to $[D_1]$. Obviously, D_1 is independent of the choice of m . Let $\Theta_{h_1} = -\sqrt{-1}\partial\bar{\partial}\log h_1$.

Let D_2 be an ample divisor of X^\dagger independent on the choice of m such that for a smooth hermitian metric h_2 , we have

$$\Theta_{h_2} = -\sqrt{-1}\partial\bar{\partial}\log h_2 > -\Theta_{h_1}.$$

Let S_2 be the defining section of D_2

Let D_3 be a divisor of X^\dagger depending on the choice of m such that $\psi_m \in C^\infty(X^\dagger \setminus D_3)$. Let S_3 be the defining section of D_3 and h_3 be a fixed smooth hermitian metric of the line bundle associated to $[D_3]$. We define $\Theta_{h_3} = -\sqrt{-1}\partial\bar{\partial}\log h_3$ and for $\epsilon > 0$ sufficiently small

$$\Theta_{h_2} > -\Theta_{h_1} - \epsilon\Theta_{h_3}.$$

Now we let

$$\psi_{m,\epsilon} = \varphi_m + \log |S_1|_{h_1}^2 + \log |S_2|_{h_2}^2 + \epsilon \log |S_3|_{h_3}^2,$$

and so $\tilde{\varphi}_{m,\epsilon}$ satisfies

$$(\chi_0 + \Theta_{h_1} + \Theta_{h_2} + \epsilon\Theta_{h_3} + \sqrt{-1}\partial\bar{\partial}\log \psi_{m,\epsilon})^\kappa \wedge \omega^{n-\kappa} = \frac{Fe^{\psi_{m,\epsilon}}}{|S_1|_{h_1}^2 |S_2|_{h_2}^2 (|S_3|_{h_3}^2)^\epsilon} \Omega_0. \quad (4.21)$$

The maximum of $\psi_{m,\epsilon}$ can only be achieved in $X^\dagger \setminus (D_1 \cup D_2 \cup D_3)$. Then by the maximum principle,

$$\sup_{X^\dagger} \psi_{m,\epsilon} \leq \sup_{X^\dagger} \left(|S_1|_{h_1}^2 |S_2|_{h_2}^2 (|S_3|_{h_3}^2)^\epsilon \frac{(\chi_0 + \Theta_{h_1} + \Theta_{h_2} + \epsilon\Theta_{h_3})^\kappa \wedge \omega^{n-\kappa}}{F\Omega_0} \right) = C_{m,\epsilon}, \quad (4.22)$$

where $\lim_{\epsilon \rightarrow \infty} C_{m,\epsilon} = C_{m,0} = \sup_{X^\dagger} \left(|S_1|_{h_1}^2 |S_2|_{h_2}^2 \frac{(\chi_0 + \Theta_{h_1} + \Theta_{h_2})^\kappa \wedge \omega^{n-\kappa}}{F\Omega_0} \right)$ and $C_{m,0}$ is independent of the choice of m .

Now let ϵ tend to 0. Then there exists a constant $C > 0$ independent of the choice of m such that

$$\sup_{X^\dagger} \psi_m \leq C,$$

that is, there exists $C' > 0$ independent of m such that

$$e^{\varphi_m} \Omega_m \leq C' |S_1|_{h_1}^{-2} |S_2|_{h_2}^{-2} \Omega_0.$$

□

Proposition 4.8 *There exists a measure Ω_{can} on X such that*

1. $\Omega_{can} = \lim_{m \rightarrow \infty} e^{\varphi_m} \Omega_m$.
2. (K_X, Ω_{can}^{-1}) is an analytic Zariski decomposition. Furthermore,

$$\frac{\Omega_{can}}{\Omega_0} < \infty, \quad \text{and} \quad \frac{\Psi_{X,\epsilon}}{\Omega_{can}} < \infty.$$

Proof By Corollary 4.3 and Lemma 4.9, we can define Ω_{can}

$$\Omega_{can} = \lim_{m \rightarrow \infty} e^{\varphi_m} \Omega_m$$

and $\frac{\Omega_m}{\Omega_0} < \infty$. Since $\varphi_m + \log \frac{\Omega_m}{\Omega_0} \in PSH(X, \chi_0)$ and $\{\varphi_m + \log \frac{\Omega_m}{\Omega_0}\}_{m=m_0}^\infty$ is convergent in $L^1(X)$. Also $PSH(X, \chi_0) \cap L^1(X)$ is closed in $L^1(X)$. Therefore

$$\lim_{m \rightarrow \infty} \left(\varphi_m + \log \frac{\Omega_m}{\Omega_0} \right) = \log \frac{\Omega_{can}}{\Omega_0}$$

in $PSH(X, \chi_0) \cap L^1(X)$ and $\log \frac{\Omega_{can}}{\Omega_0} < \infty$.

Let $h_{can} = \Omega_{can}^{-1}$ be the hermitian metric on K_X . By the construction of Ω_m ,

$$|\sigma|_{h_{can}}^2 < \infty$$

for any section $\sigma \in H^0(X, mK_X)$. Hence $\frac{\Psi_{X,\epsilon}}{\Omega_{can}} < \infty$ and

$$H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h_{can}^m)) \rightarrow H^0(X, \mathcal{O}_X(mK_X))$$

is an isomorphism and (K_X, h_{can}) is an analytic Zariski decomposition. □

Proposition 4.9 Let $\Omega^\dagger = (\pi^\dagger)^* \Omega_{can}$. There exists a closed positive $(1, 1)$ -current ω^\dagger on Y^\dagger such that $(\Phi^\dagger)^* \omega^\dagger = \sqrt{-1} \partial \bar{\partial} \log \Omega^\dagger$ on a Zariski open set of X^\dagger . Furthermore, on a Zariski open set of Y^\dagger , we have

$$(\omega^\dagger)^\kappa = (\Phi^\dagger)_* \Omega^\dagger, \tag{4.23}$$

and so

$$\text{Ric}(\omega^\dagger) = -\omega^\dagger + \omega_{WP}. \tag{4.24}$$

Proof Let $\psi_m = \varphi_m + \log \frac{\Omega_m}{\Omega_{m_0}}$. Both ψ_m and φ_m descend to Y^\dagger and by Proposition 4.8

$$\lim_{m \rightarrow \infty} \psi_m = \psi_\infty = \log \frac{\Omega^\dagger}{\Omega_{m_0}}.$$

Consider

$$\begin{array}{ccc} X_m & \xleftarrow{f_m} \cdots \cdots \cdots & X^\dagger \\ \Psi_m \downarrow & & \downarrow \Phi^\dagger \\ Y_m & \xleftarrow{g_m} \cdots \cdots \cdots & Y^\dagger \end{array}$$

For simplicity, we use ω_m for $(g_m)^*\omega_m$. Let D_{m_0} be a divisor of Y^\dagger such that on $X \setminus D_{m_0}$, ω_0 is smooth and $\log \frac{\Omega^\dagger}{\Omega_{m_0}} < \infty$. Also on $Y \setminus D_{m_0}$,

$$(\omega_{m_0} + \sqrt{-1}\partial\bar{\partial}\psi_m)^\kappa = e^{\varphi_m}(\Phi^\dagger)_*\Omega_m.$$

Since ψ_m converges uniformly on any compact set of $Y^\dagger \setminus D_{m_0}$ to ψ_∞ , we have on $Y^\dagger \setminus D_{m_0}$,

$$(\omega_{m_0} + \sqrt{-1}\partial\bar{\partial}\psi_\infty)^\kappa = \lim_{m \rightarrow \infty} (\omega_{m_0} + \sqrt{-1}\partial\bar{\partial}\log \psi_m)^n = \lim_{m \rightarrow \infty} e^{\varphi_m}(\Phi^\dagger)_*\Omega_m = (\Phi^\dagger)_*\Omega^\dagger.$$

Let $\omega^\dagger = \omega_{m_0} + \sqrt{-1}\partial\bar{\partial}\psi_\infty$. Since it is a closed positive current on $Y^\dagger \setminus D_{m_0}$ and it can be extended to a closed positive current on Y .

Also $(\Phi^\dagger)^*\omega_{m_0} = \sqrt{-1}\partial\bar{\partial}\log \Omega_{m_0}$ on $X^\dagger \setminus (\pi^\dagger)^{-1}(Bs(|m_0!K_X|))$. This implies that on $X^\dagger \setminus (\pi^\dagger)^{-1}(Bs(|m_0!K_X|))$,

$$(\Phi^\dagger)^*\omega^\dagger = \sqrt{-1}\partial\bar{\partial}\log \Omega^\dagger.$$

Furthermore, we have $\frac{\Omega_{m_0}}{\Omega^\dagger} = \frac{(\Phi^\dagger)_*\Omega_{m_0}}{(\Phi^\dagger)_*\Omega^\dagger}$ and so

$$\begin{aligned} \text{Ric}(\omega^\dagger) &= \sqrt{-1}\partial\bar{\partial}\log(\Phi^\dagger)_*\Omega^\dagger \\ &= -\omega^\dagger + \sqrt{-1}\partial\bar{\partial}\log(\Phi^\dagger)_*\Omega_{m_0} - \sqrt{-1}\partial\bar{\partial}\log \Omega_{m_0} \\ &= -\omega^\dagger + \omega_{WP}. \end{aligned}$$

□

Proposition 4.8 and Proposition 4.9 conclude the proof of Theorem B.2.

4.5 Uniqueness assuming finite generation of the canonical ring

If the canonical ring (X, R_X) is finitely generated, the canonical model X_{can} is unique and can be constructed by the pluricanonical system $|mK_X|$ for sufficiently large m . In this section, we will prove the uniqueness of the canonical measures constructed in Section 4.3 and 4.4 by assuming finite generation of the canonical ring. Furthermore, the canonical measure can be considered as a birational invariant.

Theorem 4.2 *Let X be an algebraic manifold of general type. If the canonical ring $R(X, K_X)$ is finitely generated, the Kähler-Einstein measure in Theorem B.1 is constructed in finite steps. Furthermore, it is continuous on X and smooth on a Zariski open dense set of X .*

Theorem 4.2 is an immediate consequence from the proof of Theorem B.1 with the assumption of finite generation of the canonical ring. The following theorem is proved in [EyGuZe1].

Theorem 4.3 *Let X be an algebraic manifold of general type. If the canonical ring $R(X, K_X)$ is finitely generated, X_{can} will have only canonical singularities and there exists a unique Kähler-Einstein metric ω_{can} on X_{can} with a continuous potential.*

Theorem C.1 is then proved as a corollary of Theorem 4.2 and Theorem 4.3.

Corollary 4.4 *Let X be an n -dimensional algebraic manifold of general type. Suppose that the canonical ring $R(X, K_X)$ is finitely generated and $\pi : X \dashrightarrow X_{can}$ is the pluricanonical map. Let ω_{can} be the unique Kähler-Einstein metric on X_{can} as in Theorem 4.3 and $\Omega_{KE} = \pi^*(\omega_{can}^n)$. Then (X, Ω_{KE}^{-1}) coincides with the analytic Zariski decomposition constructed in Theorem B.1.*

Proof Since the canonical ring $R(X, K_X)$ is finitely generated, the pluricanonical map is stabilized for sufficiently large power so that the proof of Theorem B.1 terminates in finite steps. It is then straightforward to check that the Kähler-Einstein metric constructed in Theorem B.1 satisfies the same Monge-Ampère equation on the unique canonical model of X in Theorem 4.3 (see [EyGuZe1]).

We shall now prove Theorem C.2.

Definition 4.2 *Suppose that X is an n -dimensional algebraic manifold of Kodaira dimension $0 < \kappa < n$ and the canonical ring $R(X, K_X)$ is finitely generated. Let $\Phi : X \dashrightarrow X_{can}$ be the pluricanonical map. There exists a nonsingular model X^\dagger of X and the following diagram holds*

$$\begin{array}{ccc} X & \xleftarrow{\pi^\dagger} & X^\dagger \\ & \searrow \Phi & \swarrow \Phi^\dagger \\ & X_{can} & \end{array} \quad (4.25)$$

where π^\dagger is barational and the generic fibre of Φ^\dagger has Kodaira dimension 0.

1. Then the pushforward measure $\Phi_*\Omega$ on X_{can} is defined by

$$\Phi_*\Omega = (\Phi^\dagger)_* ((\pi^\dagger)^*\Omega). \quad (4.26)$$

2. Let $\Phi = \Phi_m$ be the pluricanonical map associated to a basis $\{\sigma_{j_m}\}_{j_m=0}^{d_m}$ of the linear system $|mK_X|$, for m sufficiently large. Let $\Omega_m = \left(\sum_{j_m=0}^{d_m} \sigma_{j_m} \otimes \overline{\sigma_{j_m}}\right)^{\frac{1}{m}}$ and ω_{FS} be the Fubini-Study metric of \mathbb{CP}^{d_m} restricted on X_{can} associated to Φ_m . Then we defined $\bar{\omega}_{WP}$ by

$$\bar{\omega}_{WP} = \frac{1}{m} \omega_{FS} + \sqrt{-1} \partial \bar{\partial} \log \Phi_*\Omega_m. \quad (4.27)$$

In particular, $\bar{\omega}_{WP}$ coincides with ω_{WP} in Definition 4.1 on a Zariski open set of X_{can} .

Lemma 4.10 $\Phi_*\Omega$ is independent of the choice of the diagram in Definition 4.2.

Proof Let ρ be a test function on X_{can} . Then

$$\int_{X_{can}} \rho \Phi_*\Omega = \int_{X^\dagger} ((\Phi^\dagger)^*\rho) (\phi^\dagger)^*\Omega = \int_X (\Phi^*\rho) \Omega,$$

which is independent of the choice of the diagram 4.25.

□

Since the generic fibre of Φ^\dagger has Kodaira dimension 0, by the same argument in Lemma 4.5, we have the following lemma.

Lemma 4.11 *Let $\{\sigma_{j_p}^{(1)}\}_{j_p=0}^{d_p}$ and $\{\sigma_{j_q}^{(2)}\}_{j_q=0}^{d_q}$ be basis of the linear systems $|pK_X|$ and $|qK_X|$, for p and q sufficiently large. Let $\Omega^{(1)} = \left(\sum_{j_p=0}^{d_p} \sigma_{j_p} \otimes \overline{\sigma_{j_p}}\right)^{\frac{1}{p}}$ and $\Omega^{(2)} = \left(\sum_{j_q=0}^{d_q} \sigma_{j_q} \otimes \overline{\sigma_{j_q}}\right)^{\frac{1}{q}}$. Then $(\pi^\dagger)^* \left(\frac{\Omega^{(1)}}{\Omega^{(2)}}\right)$ is constant on any generic fibre and so*

$$(\pi^\dagger)^* \left(\frac{\Omega^{(1)}}{\Omega^{(2)}}\right) = (\Phi^\dagger)^* \left(\frac{\Phi_* \Omega^{(1)}}{\Phi_* \Omega^{(2)}}\right). \quad (4.28)$$

Lemma 4.12 *The definition of $\bar{\omega}_{WP}$ only depends on X .*

Proof By Lemma 4.10, the definition of Ω_{WP} does not depend on the choice of the diagram 4.13. Let $\{\sigma_{j_p}^{(1)}\}_{j_p=0}^{d_p}$ and $\{\sigma_{j_q}^{(2)}\}_{j_q=0}^{d_q}$ be basis of the linear systems $|pK_X|$ and $|qK_X|$, for p and q sufficiently large. Let $\Omega^{(1)} = \left(\sum_{j_p=0}^{d_p} \sigma_{j_p} \otimes \overline{\sigma_{j_p}}\right)^{\frac{1}{p}}$ and $\Omega^{(2)} = \left(\sum_{j_q=0}^{d_q} \sigma_{j_q} \otimes \overline{\sigma_{j_q}}\right)^{\frac{1}{q}}$.

Let $\omega_{FS}^{(1)}$ and $\omega_{FS}^{(2)}$ be the Fubini-Study metrics of \mathbf{CP}^{d_p} and \mathbf{CP}^{d_q} restricted on X_{can} associated to Φ_p and Φ_q . Then by avoiding the base locus of $R(X^\dagger, K_{X^\dagger})$, there exist a Zariski open set U of X_{can} and a Zariski open set V of X^\dagger , such that on V

$$\frac{1}{p}(\Phi^\dagger)^* \omega_{FS}^{(1)} = \sqrt{-1} \partial \bar{\partial} \log(\phi^\dagger)^* \Omega^{(1)}, \quad \frac{1}{q}(\Phi^\dagger)^* \omega_{FS}^{(2)} = \sqrt{-1} \partial \bar{\partial} \log(\phi^\dagger)^* \Omega^{(2)}$$

and so on U ,

$$\frac{1}{p} \omega_{FS}^{(1)} - \frac{1}{q} \omega_{FS}^{(2)} = \sqrt{-1} \partial \bar{\partial} \log \left(\frac{\Phi_* \Omega^{(1)}}{\Phi_* \Omega^{(2)}} \right). \quad (4.29)$$

Since $\frac{1}{p} \omega_{FS}^{(1)}$ and $\frac{1}{q} \omega_{FS}^{(2)}$ are in the same class, and $\log \left(\frac{\Phi_* \Omega^{(1)}}{\Phi_* \Omega^{(2)}} \right)$ is in $L^\infty(X_{can})$, Equation 4.29 holds everywhere on X_{can} . Therefore the following equality completes the proof of the lemma

$$\frac{1}{p} \omega_{FS}^{(1)} - \sqrt{-1} \partial \bar{\partial} \log \Phi_* \Omega^{(1)} = \frac{1}{q} \omega_{FS}^{(2)} - \sqrt{-1} \partial \bar{\partial} \log \Phi_* \Omega^{(2)}.$$

□

Theorem 4.4 *Suppose that X is an n -dimensional algebraic manifold of Kodaira dimension $0 < \kappa < n$. If the canonical ring $R(X, K_X)$ is finitely generated, X_{can} is the canonical model of X , then there exists a unique canonical measure Ω_{can} on X satisfying*

1. Ω_{can} is continuous on X and smooth on a Zariski open set of X .
2. $0 < \frac{\Omega_{can}}{\Psi_{X,M}} < \infty$ and (K_X, Ω_{can}^{-1}) is an analytic Zariski decomposition.

3. Let $\Phi : X \dashrightarrow X_{can}$ be the pluricanonical map. Then there exists a unique closed positive $(1, 1)$ -current ω_{can} with bounded local potential on X_{can} such that $\Phi^* \omega_{can} = \sqrt{-1} \partial \bar{\partial} \log \Omega_{can}$ outside the base locus of the pluricanonical system. Furthermore,

$$(\omega_{can})^\kappa = \Phi_* \Omega_{can},$$

so on a Zariski open set of X_{can} we have

$$\text{Ric}(\omega_{can}) = -\omega_{can} + \bar{\omega}_{WP}.$$

Furthermore, Ω_{can} is invariant under birational transformations.

Proof If $R(X, K_X)$ is finitely generated, there exists the following diagram

$$\begin{array}{ccc} X & \xleftarrow{\pi^\dagger} & X^\dagger \\ & \searrow \Phi & \swarrow \Phi^\dagger \\ & X_{can} & \end{array}$$

where X_{can} is the canonical model of X and X^\dagger is the resolution of the stable base locus of the pluricanonical systems such that $(\pi^\dagger)^* MK_X = L_M + E_M$ for sufficiently large M , where L_M is globally generated and E_M is the fixed part of $|(\pi^\dagger)^* MK_X|$ with E_M being a divisor with normal crossings. X^\dagger is an Iitaka fibration over X_{can} such that the generic fibre has Kodaira dimension 0. Let $\{\sigma_j\}_{j=0}^{d_M}$ be a basis of $H^0(X, MK_X)$ and $\{\zeta_j\}_{j=0}^{d_M}$ be a basis of $H^0(X, L_M)$ such that

$$(\pi^\dagger)^* \sigma_j = \zeta_j E_M.$$

Let $\Omega = \pi^\dagger \left(\sum_{m=0}^M \sum_{j_m=0}^{d_m} |\sigma_{j_m}|^{\frac{2}{m}} \right)$ be a degenerate smooth volume form on X^\dagger and $\omega = \frac{1}{M} \sqrt{-1} \partial \bar{\partial} \log \left(\sum_{j=0}^{d_M} |\zeta_j|^2 \right)$. Then the following Monge-Ampère equation has a unique continuous solution φ on X_{can}

$$(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^\kappa = e^\varphi (\Phi^\dagger)_* \Omega. \quad (4.30)$$

Furthermore, φ is smooth on a Zariski open set X_{can}° of X_{can} and so is $\omega_{can} = \omega + \sqrt{-1} \partial \bar{\partial} \varphi$.

Let $\Theta = \frac{\Omega}{(\Phi^\dagger)^* (\Phi^\dagger)_* \Omega}$ be an $(n - \kappa, n - \kappa)$ -current on X^\dagger . On a generic fibre F , $\Theta|_F = \eta \wedge \bar{\eta}$ for some $\eta \in H^0(X, K_F)$. So without loss of generality, we assume that on X_{can}° ,

$$\omega_{WP} = \sqrt{-1} \partial \bar{\partial} \log \Theta - \frac{1}{M} \sqrt{-1} \partial \bar{\partial} \log |E_M|^2.$$

Therefore on X_{can}°

$$\text{Ric}(\omega_{can}) = -\omega_{can} + \bar{\omega}_{WP}.$$

On the other hand, we define

$$\Omega_{can} = ((\pi^\dagger)^{-1})^* (e^\varphi \Omega) = e^{\Phi^* \varphi} \left(\sum_{m=0}^M \sum_{j_m=0}^{d_m} |\sigma_{j_m}|^{\frac{2}{m}} \right).$$

From the regularity of φ , Ω_{can} is continuous on X and smooth on a Zariski open set of X and

$$\sqrt{-1} \partial \bar{\partial} \log \Omega_{can} = \sqrt{-1} \partial \bar{\partial} \log \left(\sum_{m=0}^M \sum_{j_m=0}^{d_m} |\sigma_{j_m}|^{\frac{2}{m}} \right) + \sqrt{-1} \partial \bar{\partial} \Phi^* \varphi = \Phi^* \omega_{can}.$$

We then shall prove the uniqueness of Ω_{can} . Suppose there exists another measure Ω' satisfying the assumptions in the theorem. Then let $\Omega' = e^{\varphi'} \Omega$. Since $\Phi^* \omega_{FS} = \sqrt{-1} \partial \bar{\partial} \log \Omega$ and $\sqrt{-1} \partial \bar{\partial} \log \Omega' - \sqrt{-1} \partial \bar{\partial} \log \Omega$ is a pullback from X_{can} . Therefore on a Zariski open set a generic fibre F of Φ^\dagger , we have

$$\sqrt{-1} \partial_F \bar{\partial}_F \log(\pi^\dagger)^* \left(\frac{\Omega'}{\Omega} \right) = \sqrt{-1} \partial_F \bar{\partial}_F (\pi^\dagger)^* \varphi'.$$

Since $(\pi^\dagger)^* \varphi' \in L^\infty(F)$, $(\pi^\dagger)^* \varphi'$ is constant along F . So φ' descends to X_{can} and satisfies the following Monge-Ampère equation

$$(\omega + \sqrt{-1} \partial \bar{\partial} \varphi')^\kappa = e^{\varphi'} \Phi_* \Omega. \quad (4.31)$$

By the uniqueness of the solution of Equation 4.31, $\Omega' = \Omega_{can}$ and we have proved the uniqueness of Ω_{can} .

Finally we shall prove that Ω_{can} and $\Phi^* \omega_{can}$ are birational invariants. Suppose $X_{(1)}$ and $X_{(2)}$ are birational with X_{can} being the canonical model. Then we have the following diagram

$$\begin{array}{ccc} X_{(1)} & \xleftarrow{\dots \dots \dots f \dots \dots \dots} & X_{(2)} \\ & \searrow \Phi^{(1)} \quad \swarrow \Phi^{(2)} & \\ & X_{can} & \end{array}$$

where f is birational, $\Phi^{(1)}$ and $\Phi^{(2)}$ are the pluricanonical maps. Fix Ω on $X_{(1)}$ as constructed as in the proof of uniqueness, then by Hartog's theorem, $f^* \Omega$ is smooth and can be constructed the same way. By replacing $X_{(1)}$ and $X_{(2)}$ by their Iitaka fibration, it is straightforward to show that

$$(\Phi^{(1)})_* \Omega = (\Phi^{(2)})_* (f^* \Omega).$$

Let $(\Phi^{(1)})^* \omega = \sqrt{-1} \partial \bar{\partial} \log \Omega$, $\Omega_{(1)} = e^{\varphi_{(1)}} \Omega$ and $\Omega_{(2)} = e^{\varphi_{(2)}} f^* \Omega$ be the unique canonical measures on $X_{(1)}$ and $X_{(2)}$. Both $\varphi_{(1)}$ and $\varphi_{(2)}$ descend to $PSH(X, \omega) \cap L^\infty(X_{can})$ and satisfy

$$(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^\kappa = e^\varphi (\Phi^{(1)})_* \Omega.$$

The uniqueness of the solution to the above Monge-Ampère equation implies that $\varphi_{(1)} = \varphi_{(2)}$ and so

$$f^* \Omega_{(1)} = \Omega_{(2)}.$$

□

5 The Kähler-Ricci flow

5.1 Reduction of the normalized Kähler-Ricci flow

Let X be an n -dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form ω on X . In local coordinates z_1, \dots, z_n , we can write ω as

$$\omega = \sqrt{-1} \sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge d\bar{z}_{\bar{j}},$$

where $\{g_{i\bar{j}}\}$ is a positive definite hermitian matrix function. Consider the normalized Kähler-Ricci flow

$$\begin{cases} \frac{\partial \omega(t, \cdot)}{\partial t} = -\text{Ric}(\omega(t, \cdot)) - \omega(t, \cdot), \\ \omega(0, \cdot) = \omega_0. \end{cases} \quad (5.1)$$

where $\text{Ric}(\omega(t, \cdot))$ denotes the Ricci curvature of $\omega(t, \cdot)$ and ω_0 is a given Kähler metrics.

Let $Ka(X)$ denote the Kähler cone of X , that is,

$$Ka(X) = \{[\omega] \in H^{1,1}(X, \mathbf{R}) \mid [\omega] > 0\}.$$

Suppose that $\omega(t, \cdot)$ is a solution of (5.1) on $[0, T)$. Then its induced equation on Kähler classes in $Ka(X)$ is given by the following ordinary differential equation

$$\begin{cases} \frac{\partial [\omega]}{\partial t} = -2\pi c_1(X) - [\omega] \\ [\omega]|_{t=0} = [\omega_0]. \end{cases} \quad (5.2)$$

It follows

$$[\omega(t, \cdot)] = -2\pi c_1(X) + e^{-t}([\omega_0] + 2\pi c_1(X)).$$

Now if we assume that canonical bundle K_X is semi-positive, then for a sufficiently large integer m , the pluricanonical map associated to $H^0(X, mK_X)$ gives rise to an algebraic fibre space $f : X \rightarrow X_{can}$, where X_{can} is the canonical model of X . Recall the Kodaira dimension $kod(X)$ of X is defined to be the dimension of X_{can} . Moreover, there is a smooth Kähler form χ as the Fubini-Study metric associated to a basis of $H^0(X, mK_X)$ on the normal Kähler space X_{can} such that $f^*\chi$ represents $-2\pi c_1(X)$. Choose the reference Kähler metric ω_t by

$$\omega_t = \chi + e^{-t}(\omega_0 - \chi). \quad (5.3)$$

Here we abuse the notation by identifying χ and $f^*\chi$ for simplicity. Then the solution of (5.1) can be written as

$$\omega = \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi.$$

We can always choose a smooth volume form Ω on X such that $\text{Ric}(\Omega) = \chi$. Then the evolution for the Kähler potential φ is given by the following initial value problem:

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{e^{(n-\kappa)t}(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} - \varphi \\ \varphi|_{t=0} = 0, \end{cases} \quad (5.4)$$

where $\kappa = \text{cod}(X)$.

5.2 Kähler-Ricci flow on algebraic manifolds with semi-positive canonical line bundle

Theorem 5.1 *Let X be a nonsingular algebraic variety with semi ample canonical line bundle K_X and so X admits a holomorphic fibration over its canonical model X_{can} $f : X \rightarrow X_{\text{can}}$. Then for any initial Kähler metric, the Kähler-Ricci flow (1.1) has a global solution $\omega(t, \cdot)$ for all time $t \in [0, \infty)$ satisfying:*

1. $\omega(t, \cdot)$ converges to $f^*\omega_\infty \in -2\pi c_1(X)$ as currents for a positive current ω_∞ on Σ .
2. ω_∞ is smooth on X_{can}° and satisfies the generalized Kähler-Einstein equation on X_{can}°

$$\text{Ric}(\omega_\infty) = -\omega_\infty + \omega_{WP}, \quad (5.5)$$

where ω_{WP} is the induced Weil-Petersson metric.

3. for any compact subset $K \in X_{\text{reg}}$, there is a constant C_K such that

$$\|R(t, \cdot)\|_{L^\infty(K)} + e^{(n-\kappa)t} \sup_{f^{-1}(s) \in K} \|(\omega(t, \cdot))^{n-\kappa}|_{f^{-1}(s)}\|_{L^\infty} \leq C_K. \quad (5.6)$$

Corollary 5.1 *Let X be a nonsingular algebraic variety with semi-ample canonical bundle. If $X_{\text{can}}^\circ = X_{\text{can}}$, i.e., X_{can} is nonsingular and $f : X \rightarrow X_{\text{can}}$ has no singular fibres, then for any initial Kähler metric, the Kähler-Ricci flow (1.1) converges to a smooth limit metric $f^*\omega_\infty \in K_X$ satisfying*

$$\text{Ric}(\omega_\infty) = -\omega_\infty + \omega_{WP}. \quad (5.7)$$

Step 1. Zeroth order and volume estimates

We will first derive the zeroth order estimates for φ and $\frac{d\varphi}{dt}$.

Lemma 5.1 *Let φ be a solution of the Kähler-Ricci flow (5.4). There exists a constant $C > 0$ such that on $[0, \infty) \times X$*

1. $\varphi \leq C$,
2. $\frac{\partial \varphi}{\partial t} \leq C$,
3. $\frac{e^{(n-\kappa)t} \omega^n}{\Omega} \leq C$.

Proof The lemma is a straightforward application of the maximum principle and can be proved by the same argument as in [SoTi]. □

Proposition 5.1 *There exists a constant $C > 0$ such that on $[0, \infty) \times X$*

$$|\varphi| \leq C. \quad (5.8)$$

Proof Rewrite the parabolic flow as a family of Monge-Ampère equations

$$(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{\frac{\partial\varphi}{\partial t} + \varphi - (n-\kappa)t}\Omega.$$

We will apply Theorem 2.5 by letting $F(t, \cdot) = e^{\frac{\partial\varphi}{\partial t} + \varphi - (n-\kappa)t}$. Notice that there exists a constant $C_1 > 0$ such that $0 < F \leq C_1 e^{-(n-\kappa)t}$,

$$\frac{e^{-(n-\kappa)t}}{C_1} \leq [\omega_t]^n \leq C_1 e^{-(n-\kappa)t}$$

and

$$\frac{e^{-(n-\kappa)t}}{C_1} \chi^\kappa \wedge \omega_0^{n-\kappa} \leq \omega_t^n \leq C_1 e^{-(n-\kappa)t} \Omega.$$

The assumptions in Theorem 2.5 for F and ω_t are satisfied. Therefore $\sup_X \varphi - \inf_X \varphi$ is uniformly bounded for all $t \in [0, \infty)$. Since φ is uniformly bounded from above, the proposition is proved and the uniform C^0 -estimate is obtained. \square

The following estimate can be proved in the same way as in [SoTi].

Lemma 5.2 *There exists a divisor D on X_{can} and constants C_1 and $C_2 > 0$ such that*

$$\frac{\partial\varphi}{\partial t} \geq C_1 \log |S|_h^2 - C_2, \quad (5.9)$$

where S is a defining section of f^*D and h is a fixed smooth hermitian metric of the line bundle associated to $[f^*D]$.

Step 2. Partial second order estimates and collapsing

Proposition 5.2 *There exist a divisor D on X_{can} and constants $\lambda, C > 0$ such that*

$$\mathrm{tr}_\omega(\chi) \leq \frac{C}{|S|_h^{2\lambda}}, \quad (5.10)$$

where S is a defining section of f^*D and h is smooth hermitian metric of the line bundle associated to the divisor $[f^*D]$.

Proof Since X_{can} might be singular, we can consider the nonsingular model $f' : X' \rightarrow Y'$ for $f : X \rightarrow X_{can}$ such that following diagram commutes

$$\begin{array}{ccc} X & \xleftarrow{\pi} & X' \\ f \downarrow & & \downarrow f' \\ X_{can} & \xleftarrow{\mu} & Y' \end{array}$$

where π and μ are birational.

Let $\varphi' = \pi^*\varphi$, $\chi' = \mu^*\chi$, $\omega'_0 = \pi^*\omega$ and $\omega' = \pi^*\omega$. We also write χ' for $(f')^*\chi'$ for simplicity.

Let θ be a Kähler form on Y' such that $\theta \geq \chi'$. For simplicity, we identify θ and χ' with $(f')^*\theta$ and $(f')^*\chi'$. Since χ' is semi-positive induced by the Fubini-Study metric and it only vanishes along a subvariety of Y' with finite order, there exists a divisor D_1 on Y' such that

$$\theta \leq \frac{1}{|S_1|_{h_1}^2} \chi',$$

where S_1 is a defining section of $(f')^*D_1$ with h_1 a smooth hermitian metric of the line bundle associated to $[(f')^*D_1]$. Without loss of generality, we can assume that the support of μ^*D in Lemma 5.2 is contained in D_1 .

Let

$$u = \text{tr}_{g'}(\theta) = (g')^{i\bar{j}} \theta_{i\bar{j}},$$

where g' is the Kähler metric associated to ω' . The Kähler-Ricci flow for ω can be pulled back to the Kähler-Ricci flow for ω' on X' outside the exceptional divisors. Let Δ' be the Laplace operator associated to g' . We have then

$$\text{tr}_{g'}(\theta) \leq \frac{1}{|S_1|_{h_1}^2} \text{tr}_{g'}(\chi').$$

Following the similar calculation in [SoTi], we have

$$\left(\frac{\partial}{\partial t} - \Delta' \right) \log u \leq C(u + 1) \quad (5.11)$$

Since $[\chi']$ is big and semi-ample, there exists a divisor D_2 on Y' such that $[\chi'] - \epsilon[D_2]$ is ample for any $\epsilon > 0$. Then let S_2 be the defining section for $(f')^*D_2$ and there exists a smooth hermitian metric h_2 on the line bundle associated to $[D_2]$ such that

$$\chi' - \epsilon\Theta_{h_2} = \chi' + \epsilon\sqrt{-1}\partial\bar{\partial} \log h_2 > 0.$$

For simplicity, we identify h_2 and $(f')^*h_2$.

Let D_3 be a divisor on X' containing the exceptional divisor of π on X' . Let S_3 be the defining section of $[D_3]$. There exists a smooth hermitian metric h_3 on the line bundle associated to $[D_3]$ such that for all sufficiently small $\delta > 0$

$$\omega'_0 + \delta\Theta_{h_3} \geq 0.$$

We define

$$\varphi'_\epsilon = \varphi' - \epsilon \log |S_2|_{h_2}^2.$$

Then there exists a constant $C > 0$ depending on ϵ such that

$$\Delta' \varphi'_\epsilon = n - \text{tr}_{g'}(\omega'_t - \epsilon\Theta_{h_2}) \leq n - C\text{tr}_{g'}\theta - e^{-t}\text{tr}_{g'}(\omega'_0) = n - Cu - e^{-t}\text{tr}_{g'}(\omega'_0).$$

Calculate for sufficiently large $A > 0$ and small $\epsilon > 0$ and $\delta > 0$

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta' \right) \left(\log \left(|S_1|_{h_1}^4 |S_3|_{h_3}^{2\delta e^{-t}} u \right) - 2A\varphi'_\epsilon \right) \\ & \leq -Au + C - \delta e^{-t} \log |S_3|_{h_3}^2 - 2A \frac{\partial \varphi'}{\partial t}. \end{aligned}$$

for all $t > 0$ in $X' \setminus (D_1 \cup D_2 \cup D_3)$.

The maximum of $\log \left(|S_1|_{h_1}^4 |S_3|_{h_3}^{2\delta e^{-t}} u \right) - 2A\varphi'_\epsilon$ can only be achieved on $X' \setminus (D_1 \cup D_2 \cup D_3)$. The maximum principle implies that there exist constants $\lambda, C > 0$ independent of δ such that for all $(t, z) \in [0, \infty) \times X'$

$$u(t, z) \leq C \left(|S_1|_{h_1}^{-2\lambda} |S_2|_{h_2}^{-2\lambda} |S_3|_{h_3}^{-2\delta e^{-t}} \right) (z).$$

The proposition is then proved by letting $\delta \rightarrow 0$. □

From the uniform upper bound of ω^n , one immediately concludes that the volume of a regular fibre of f tends to 0 exponentially fast uniformly away from the singular fibres.

Corollary 5.2 *There exists a divisor D on X_{can} and constants $\lambda, C > 0$ such that for all $t \geq 0$ and $s \in X_{can}$*

$$\frac{(\omega|_{X_s})^{n-\kappa}}{(\omega_0|_{X_s})^{n-\kappa}} \leq e^{-(n-\kappa)t} \frac{C}{|S|_h^2}, \quad (5.12)$$

where $\omega_0|_{X_s}$ and $\omega|_{X_s}$ are the restriction of ω_0 and ω_s on $X_s = f^{-1}(s)$, S is a defining section of f^*D and h is smooth hermitian metric of the line bundle associated to the divisor $[f^*D]$.

Proof Notice that

$$\frac{(\omega|_{X_s})^{n-\kappa}}{(\omega_0|_{X_s})^{n-\kappa}} = \frac{\omega^{n-\kappa} \wedge \chi^\kappa}{\omega_0^{n-\kappa} \wedge \chi^\kappa} = \frac{\omega^{n-\kappa} \wedge \chi^\kappa}{\omega^n} \frac{\omega^n}{\omega_0^{n-\kappa} \wedge \chi^\kappa} \leq C \left(\frac{\omega^{n-1} \wedge \chi}{\omega^n} \right)^\kappa \frac{\omega^n}{\omega_0^{n-\kappa} \wedge \chi^\kappa}.$$

The corollary is then proved by Lemma 5.1 and Proposition 5.2. □

Corollary 5.3 *For any compact set $K \subset X_{can}^\circ$, there exists a constant C_K such that for all $t \geq 0$ and $s \in K$*

$$\sup_{X_s} \varphi(t, \cdot) - \inf_{X_s} \varphi(t, \cdot) \leq C_K e^{-t}. \quad (5.13)$$

Proof The Poincare and Sobolev constants with respect to $\omega_0|_{X_s}$ for $s \in K$ are uniformly bounded. The proof of the corollary is achieved by Corollary 5.2 and Moser's iteration in Yau's C^0 -estimate for the Calabi conjecture. □

Step 3. Gradient estimates The gradient estimates in this section are obtained in the same way as in [SoTi] and it is an adaption from the gradient estimate in [ChYa] and the argument in [Pe] to obtain a uniform bound for $|\nabla \frac{\partial \varphi}{\partial t}|_g$ and the scalar curvature R . Let $u = \frac{\partial \varphi}{\partial t} + \varphi = \log \frac{e^{(n-\kappa)t}\omega^2}{\Omega}$. The evolution equation for u is given by

$$\frac{\partial u}{\partial t} = \Delta u + \text{tr}_\omega(\chi) - (n - \kappa). \quad (5.14)$$

We will obtain a gradient estimate for u , which will help us bound the scalar curvature from below. Note that u is uniformly bounded from above, so we can find a constant $A > 0$ such that $A - u \geq 1$.

Theorem 5.2 *There exist constants $\lambda, C > 0$ such that*

1. $|S|_h^{2\lambda} |\nabla u|^2 \leq C(A - u),$
2. $-|S|_h^{2\lambda} \Delta u \leq C(A - u),$

where ∇ is the gradient operator with respect to the metric g associated to ω along the flow and $|\cdot| = |\cdot|_g$.

Theorem 5.2 is proved the same way as in [SoTi] with little modification. The following corollary is immediate by Theorem 5.2, Lemma 5.1 and Lemma 5.2.

Corollary 5.4 *For any $\delta > 0$, there exist constants $\lambda, C > 0$ such that*

1. $|S|_h^{2\lambda} |\nabla u|^2 \leq C,$
2. $-|S|_h^{2\lambda} \Delta u \leq C.$

Now we are in the position to prove a uniform bound for the scalar curvature. The following corollary tells that the Kähler-Ricci flow will collapse with bounded scalar curvature away from the singular fibres.

Corollary 5.5 *Along the Kähler-Ricci flow (1.1) the scalar curvature R is uniformly bounded on any compact subset of X_{reg} . More precisely, there exist constants $\lambda, C > 0$ such that*

$$-C \leq R \leq \frac{C}{|S|_h^{2\lambda}}. \quad (5.15)$$

Proof It suffices to give an upper bound for R since the scalar curvature R is uniformly bounded from below by the maximum principle (cf. [SoTi]). Notice that $R_{i\bar{j}} = -u_{i\bar{j}} - \chi_{i\bar{j}}$ and then

$$R = -\Delta u - \text{tr}_\omega(\chi).$$

By Corollary 5.4 and the partial second order estimate, there exist constants $\lambda_6, C > 0$ such that

$$R \leq \frac{C}{|S|_h^{2\lambda}}.$$

□

Step 4. Uniform convergence

Let φ_∞ be the unique solution solving equation (3.12) in Theorem 3.2. We identify $f^*\varphi_\infty$ and φ_∞ for convenience.

Since K_X is semi-ample, there exists an ample line bundle L on X_{can} such that $K_X = f^*L = \frac{1}{m}f^*\mathcal{O}(1)$ for a fixed pluricanonical map. Let D be an ample divisor on X_{can} such that $[D] = \mu[L]$ for a sufficiently large integer μ , $X_{can} \setminus X_{can}^\circ \subset D$ and $\varphi_\infty \in C^\infty(X_{can} \setminus D)$. Let S_D be the defining section of D . Let h_{FS} be the Fubini study metric on $\mathcal{O}(1)$ induced by the pluricanonical map. Then there exists a continuous hermitian metric $h_D = (h_{FS})^{\frac{\mu}{m}} e^{-\mu\varphi_\infty}$ on L^μ such that $-\sqrt{-1}\partial\bar{\partial}\log h_D = \mu\chi_\infty$ since φ_∞ is continuous.

We define

$$B_r(D) = \{y \in X_{can} \mid \text{dist}_\chi(y, D) \leq r\}$$

be the geodesic tubular neighborhood of D with respect to χ and we let $\mathcal{B}_r(D) = f^{-1}(B_r(D))$.

Since φ_∞ is bounded on X and φ is uniformly bounded from above. Therefore for any $\epsilon > 0$, there exists $r_\epsilon > 0$ with $\lim_{\epsilon \rightarrow 0} r_\epsilon = 0$, such that for any $z \in \mathcal{B}_{r_\epsilon}(D)$ and $t \geq 0$ we have

$$(\varphi - \varphi_\infty + \epsilon \log |S_D|_{h_D}^2)(t, z) < -1$$

and

$$(\varphi - \varphi_\infty - \epsilon \log |S_D|_{h_D}^2)(t, z) > 1$$

Let η_ϵ be a smooth cut off function on X_{can} such that $\eta_\epsilon = 1$ on $X_{can} \setminus B_{r_\epsilon}(D)$ and $\eta_\epsilon = 0$ on $B_{\frac{r_\epsilon}{2}}(D)$.

Suppose the semi-flat closed form is given by $\omega_{SF} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\rho_{SF}$ and ρ_{SF} blows up near the singular fibres. We let ρ_ϵ be an approximation for ρ_{SF} given by

$$\rho_\epsilon = (f^*\eta_\epsilon) \rho_{SF}.$$

We also define $\omega_{SF, \epsilon} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\rho_\epsilon$. Now we define the twisted difference of φ and φ_∞ by

$$\psi_\epsilon^- = \varphi - (1 + \epsilon)\varphi_\infty - e^{-t}\rho_\epsilon + \epsilon \log |S_D|_{h_D}^2$$

and

$$\psi_\epsilon^+ = \varphi - (1 - \epsilon)\varphi_\infty - e^{-t}\rho_\epsilon - \epsilon \log |S_D|_{h_D}^2$$

where S_D is the defining section of D and h is a fixed smooth hermitian metric of the line bundle induced by $[D]$. We identify $f^*(|S_D|_{h_D}^2)$ and $|S_D|_{h_D}^2$ for convenience.

Proposition 5.3 *There exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, there exists $T_\epsilon > 0$ such that for any $z \in X$ and $t > T_\epsilon$ we have*

$$\psi_\epsilon^-(t, z) \leq 2\epsilon. \tag{5.16}$$

and

$$\psi_\epsilon^+(t, z) \geq -2\epsilon. \tag{5.17}$$

Proof The evolution equation for ψ_ϵ^- is given by

$$\frac{\partial \psi_\epsilon^-}{\partial t} = \log \frac{e^{(n-\kappa)t}((1 + \epsilon\mu - e^{-t})\chi_\infty + e^{-t}\omega_{SF,\epsilon} + \sqrt{-1}\partial\bar{\partial}\psi_\epsilon^-)^n}{C_{n,\kappa}\chi_\infty^\kappa \wedge \omega_{SF}^{n-\kappa}} - \psi_\epsilon^- - \epsilon\varphi_\infty + \epsilon \log |S|_h^2. \quad (5.18)$$

Since ρ_ϵ is bounded on X , we can always choose $T_1 > 0$ sufficiently large such that for $t > T_1$

1. $\psi_\epsilon^-(t, z) < -\frac{1}{2}$ on $\mathcal{B}_{r_\epsilon}(D)$,
2. $\sum_{p=0}^{\kappa-1} C_{n,p} e^{-(n-p)t} \left| \frac{\chi_\infty^p \wedge \omega_{SF}^{n-p}}{\chi_\infty^\kappa \wedge \omega_{SF}^{n-\kappa}} \right| \leq \epsilon$ on $X \setminus \mathcal{B}_{r_\epsilon}(D)$.

We will discuss in two cases for $t > T_1$.

1. If $\psi_{\epsilon, \max}^-(t) = \max_X \psi_\epsilon^-(t, \cdot) = \psi_\epsilon^-(t, z_{\max, t}) > 0$ for all $t > T_1$. Then $z_{\max, t} \in X \setminus \mathcal{B}_{r_\epsilon}(D)$ for all $t > T_1$ and so $\omega_{SF, \epsilon}(z_{\max, t}) = \omega_{SF}(z_{\max, t})$. Applying the maximum principle at $z_{\max, t}$, we have

$$\begin{aligned} & \frac{\partial \psi_\epsilon^-}{\partial t}(t, z_{\max, t}) \\ & \leq \left(\log \frac{e^{(n-\kappa)t}((1 + \epsilon\mu - e^{-t})\chi_\infty - \epsilon\varphi_\infty + e^{-t}\omega_{SF,\epsilon})^n}{C_{n,\kappa}\chi_\infty^\kappa \wedge \omega_{SF}^{n-\kappa}} - \psi_\epsilon^- - \epsilon\varphi_\infty + \epsilon \log |S|_h^2 \right)(t, z_{\max, t}) \\ & = \left(\log \frac{\sum_{p=0}^{\kappa} \binom{n}{\kappa} (1 + \epsilon\mu - e^{-t})^p \chi_\infty^p \wedge \omega_{SF,\epsilon}^{n-p}}{\binom{n}{\kappa} \chi_\infty^\kappa \wedge \omega_{SF}^{n-\kappa}} - \psi_\epsilon^- - \epsilon\varphi_\infty + \epsilon \log |S|_h^2 \right)(t, z_{\max, t}) \\ & \leq -\psi_\epsilon^-(t, z_{\max, t}) + \log(1 + (\mathcal{A} + 1)\epsilon) + \epsilon. \end{aligned}$$

Applying the maximum principle again, we have

$$\psi_\epsilon^- \leq (\mathcal{A} + 2)\epsilon + O(e^{-t}) \leq (\mathcal{A} + 3)\epsilon, \quad (5.19)$$

if we choose ϵ sufficiently small in the beginning and then t sufficiently large.

2. If there exists $t_0 \geq T_1$ such that $\max_{z \in X} \psi_\epsilon^-(t_0, z) = \psi_\epsilon^-(t_0, z_0) < 0$ for some $z_0 \in X$. Assume t_1 is the first time when $\max_{z \in X, t \leq t_1} \psi_\epsilon^-(t, z) = \psi_\epsilon^-(t_1, z_1) \geq (\mathcal{A} + 3)\epsilon$. Then $z_1 \in X \setminus \mathcal{B}_{r_\epsilon}(D)$ and applying the maximum principle we have

$$\begin{aligned} \psi_\epsilon^-(t_1, z_1) & \leq \left(\log \frac{e^{(n-\kappa)t}((1 + \epsilon\mu - e^{-t})\chi_\infty + e^{-t}\omega_{SF,\epsilon})^n}{\binom{n}{\kappa} \chi_\infty^\kappa \wedge \omega_{SF}^{n-\kappa}} - \psi_\epsilon^- - \epsilon\varphi_\infty + \epsilon \log |S|_h^2 \right)(t_1, z_1) \\ & \leq \log(1 + (\mathcal{A} + 1)\epsilon) + \epsilon < (\mathcal{A} + 2)\epsilon. \end{aligned}$$

which contradicts the assumption that $\psi_\epsilon^-(t_1, z_1) \geq (\mathcal{A} + 3)\epsilon$. Hence we have

$$\psi_\epsilon^- \leq (\mathcal{A} + 3)\epsilon.$$

By the same argument we have

$$\psi_\epsilon^+ \geq -(\mathcal{A} + 3)\epsilon.$$

This completes the proof. \square

Proposition 5.4 *On any compact set K of $X \setminus D$, we have*

$$\lim_{t \rightarrow \infty} \|\varphi(t, \cdot) - \varphi_\infty(\cdot)\|_{C^0(K)} = 0. \quad (5.20)$$

Proof By Proposition 5.3, we have for $t > T_\epsilon$

$$\varphi_\infty(t, z) + \epsilon \log |S|_h^2(t, z) - 3\epsilon \leq \varphi(t, z) \leq \varphi_\infty(t, z) - \epsilon \log |S|_h^2(t, z) + 3\epsilon.$$

Then the proposition is proved by letting $\epsilon \rightarrow 0$. \square

5.3 Kähler-Ricci flow and minimal model program

The Kähler-Ricci flow on algebraic manifolds of positive Kodaira dimension seems to be closely related to the minimal model program in algebraic geometry.

For any nonsingular minimal model X of positive Kodaira dimension, the canonical line bundle K_X is nef and so the Kähler-Ricci flow (5.1) has long time existence [TiZha]. The abundance conjecture predicts that K_X is semi-ample, hence the canonical ring of X is finitely generated. If we assume the abundance conjecture, the Kähler-Ricci flow will converge to the unique canonical metric on the canonical model X_{can} associated to X for any initial Kähler metric by Theorem A.

If X is not minimal, the Kähler-Ricci flow (5.1) will develop finite time singularities. Let T_1 be the first time such that $e^{-t}[\omega_0] - (1 - e^{-t})2\pi c_1(X)$ fails to be a Kähler class. Adopting arguments in [TiZha], one can show that there is a unique limiting current $\omega_{T_1}(\cdot) = \lim_{t \rightarrow T_1^-} \omega(t, \cdot) \in e^{-T_1}[\omega_0] - (1 - e^{-T_1})2\pi c_1(X)$ and it is smooth outside an analytic subvariety of X . Furthermore, the local potential φ_{T_1} of ω_{T_1} is continuous. We conjecture that X_1 , the metric completion of ω_{T_1} , is again an algebraic variety and X_1 can be obtained by certain standard algebraic procedure such as a blow-down or flip. It is reasonable to expect that such a variety X_1 does not have too bad singularities. In particular, we expect that a weak Kähler-Ricci flow can be defined on X_1 . Suppose this is true, we hope that the above procedure can be repeated as long as the canonical line bundle is not nef. We further conjecture that after repeating the above process finitely many times, we obtain the metric completions X_1, X_2, \dots, X_N such that K_{X_N} is nef ! Consequently, X_N is a minimal model of X .

It provides a new understanding of the minimal model program from an analytic point of view. We believe that it is interesting to further explore this connection between the minimal model program and the study of the regularity and convergence problem of the Kähler-Ricci flow on algebraic varieties.

6 Adjunction formulas for energy functionals

6.1 Generalized constant scalar curvature Kähler metrics

In fact, the canonical metrics in Section 3 belong to a class of Kähler metrics which generalize Calabi's extremal metrics. Let Y be a Kähler manifold of complex dimension n together with a fixed closed $(1,1)$ -form θ . Fix a Kähler class $[\omega]$, denote by $\mathcal{K}_{[\omega]}$ the space of Kähler metrics within the same Kähler class, that is, all Kähler metrics of the form $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$. One may consider the following equation:

$$\bar{\partial}V_\varphi = 0, \quad (6.1)$$

where V_φ is defined by

$$\omega_\varphi(V_\varphi, \cdot) = \bar{\partial}(S(\omega_\varphi) - \text{tr}_{\omega_\varphi}(\theta)). \quad (6.2)$$

Clearly, when $\theta = 0$, (6.1) is exactly the equation for Calabi's extremal metrics. For this reason, we call a solution of (6.1) a generalized extremal metric. If Y does not admit any nontrivial holomorphic vector fields, then any generalized extremal metric ω_φ satisfies

$$S(\omega_\varphi) - \text{tr}_{\omega_\varphi}(\theta) = \mu, \quad (6.3)$$

where μ is the constant given by

$$\mu = \frac{(2\pi c_1(Y) - [\theta]) \cdot [\omega]^{n-1}}{[\omega]^n}.$$

Moreover, if $2\pi c_1(Y) - [\theta] = \lambda[\omega]$, then any such a metric satisfies

$$\text{Ric}(\omega_\varphi) = \lambda\omega_\varphi + \theta,$$

that is, ω_φ is a generalized Kähler-Einstein metric. This can be proved by an easy application of the Hodge theory. More interestingly, if we take θ to be the pull-back of ω_{WP} by $f : X_{can}^\circ \rightarrow \mathcal{M}_{CY}$, then we get back those generalized Kähler-Einstein metrics which arise from limits of the Kähler-Ricci flow.

Let $f : X \rightarrow \Sigma$ be a Kähler surface admitting a non-singular holomorphic fibration over a Riemann surface Σ of genus greater than one, with fibres of genus at least 2. Let V be the vertical tangent bundle of X and $[\omega_t] = -f^*c_1(\Sigma) - tc_1(V)$.

Let χ be a Kähler form in $-c_1(\Sigma)$ and $\omega_0 \in -c_1(V)$. Then $\omega_0 = \omega_H \oplus \theta\chi$, where ω_H is the hyperbolic Kähler form on each fiber and θ is a smooth function on X . We then set

$$\omega_t = \chi + t\omega_0.$$

The following theorem is proved by Fine in [Fi].

Theorem 6.1 *For $t > 0$ sufficiently small, there exists a constant scalar curvature Kähler metric in $[\omega_t]$. Furthermore, such a family of constant scalar curvature Kähler metrics converge to a Kähler metric χ_∞ on Σ defined by*

$$S(\chi_\infty) - \text{tr}_{\chi_\infty}(\theta) = \text{const.} \quad (6.4)$$

where θ is the Weil-Petersson metric pulled back from the moduli spaces of the fibre curves.

6.2 Asymptotics of the Mabuchi energy by the large Kähler structure limits

Let X be an n -dimensional compact Kähler manifold and ω a Kähler form. The Mabuchi energy functional $\mathcal{K}_\omega(\cdot)$ is defined on $PSH(X, \omega)$ as follows

$$\mathcal{K}_\omega(\varphi) = \int_X \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n - \sum_{j=0}^{n-1} \int_X \varphi \operatorname{Ric}(\omega) \wedge \omega^j \wedge \omega_\varphi^{n-j-1} + \frac{n\mu}{n+1} \sum_{j=0}^n \int_X \varphi \omega^j \wedge \omega_\varphi^{n-j}, \quad (6.5)$$

where $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ and $\mu = \frac{2\pi c_1(X) \cdot [\omega]^{n-1}}{[\omega]^n}$.

Definition 6.1 *Let X be a compact Kähler manifold of complex dimension n . Let ω be a Kähler metric and θ a closed $(1, 1)$ -form on X . Then the generalized Mabuchi energy functional $\mathcal{K}_{\omega, \theta}(\cdot)$ is defined by*

$$\mathcal{K}_{\omega, \theta}(\varphi) = \int_X \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n - \sum_{j=0}^{n-1} \int_X \varphi (\operatorname{Ric}(\omega) - \theta) \wedge \omega^j \wedge \omega_\varphi^{n-j-1} + \frac{n\mu}{n+1} \sum_{j=0}^n \int_X \varphi \omega^j \wedge \omega_\varphi^{n-j}, \quad (6.6)$$

where $\mu = \frac{(2\pi c_1(X) - [\theta]) \cdot [\omega]^{n-1}}{[\omega]^n}$.

The following proposition can be proved by straightforward calculation.

Proposition 6.1

$$\delta \mathcal{K}_{\omega, \theta} = - \int_X \delta \varphi (S(\omega_\varphi) - \operatorname{tr}_{\omega_\varphi}(\theta)) \omega_\varphi^n. \quad (6.7)$$

Therefore

$$\mathcal{K}_{\omega, \theta}(\varphi) = - \int_0^1 \int_X \dot{\varphi}_t (S(\omega_t) - \operatorname{tr}_{\omega_t}(\theta)) \omega_t^n dt \quad (6.8)$$

where $\{\varphi_t\}_{t \in [0, 1]}$ is a smooth path in $PSH(X, \omega)$ with $\varphi_0 = 0$ and $\varphi_1 = \varphi$, and $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t$. The formula 6.8 is independent of the choice of the path φ_t .

Let X be an n -dimensional compact Kähler manifold with semi-ample canonical line bundle. Suppose $0 < \operatorname{kod}(X) = \kappa < n$ and $X_{can} = X_{can}^\circ$, i.e., X_{can} is nonsingular and the algebraic fibration $f : X \rightarrow X_{can}$ has no singular fibre.

Fix $\chi \in -c_1(X)$ as in Section 3.1 and ω_0 an arbitrary Kähler form with $\int_{X_s} \omega_{0,s}^{n-\kappa} = 1$, where $X_s = f^{-1}(s)$ and $\omega_{0,s} = \omega_0|_{X_s}$. Let $\omega_t = \chi + t\omega_0$ and $\omega_\varphi = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi$. Let ω_{SF} be the semi-flat form in $[\omega_0]$. Then φ can always be decomposed as

$$\varphi = \bar{\varphi} + t\psi$$

where $\bar{\varphi} = \int_{X_y} \varphi \omega_{SF}^{n-k}$ is the push-forward of φ with respect to the flat metric on the fibres.

Theorem 6.2 *Along the above class deformation of the Kähler class on X ,*

$$\mathcal{K}_{\omega_t}(\varphi) = \binom{n}{\kappa} t^{n-\kappa} (\mathcal{K}_{\chi, \omega_{WP}}(\bar{\varphi}) + \mathcal{L}_{\chi, \chi\bar{\varphi}, \omega_0}(\psi)) + O(t^{n-\kappa+1}) \quad (6.9)$$

and

$$\begin{aligned} \mathcal{L}_{\chi, \chi_{\bar{\varphi}}, \omega_0}(\psi) &= \int_{s \in X_{can}} \left(\int_{X_s} \log \frac{\omega_{\psi, s}^{n-\kappa}}{\omega_{0, s}^{n-\kappa}} \omega_{\psi, s}^{n-\kappa} \right) \chi_{\bar{\varphi}}^{\kappa} \\ &\quad - \sum_{j=0}^{n-\kappa-1} \sum_{i=0}^{\kappa} A_{i,j} \int_{s \in X_{can}} \left(\int_{X_s} \psi \text{Ric}(\omega_{0,s}) \wedge \omega_{0,s}^j \wedge \omega_{\psi,s}^{n-\kappa-1-j} \right) \chi^i \wedge \chi_{\bar{\varphi}}^{\kappa-i}, \end{aligned} \quad (6.10)$$

where $A_{i,j} = \binom{n}{\kappa}^{-1} \binom{i+j}{i} \binom{n-1-i-j}{\kappa-i}$ and $\mathcal{K}_{\chi, \omega_{WP}}(\cdot)$ is the generalized Mabuchi energy on X_{can} .

In particular, when $\chi = \chi_{\bar{\varphi}}$,

$$\mathcal{L}_{\chi, \chi_{\bar{\varphi}}, \omega_0}(\psi) = \int_{s \in X_{can}} \mathcal{K}_{\omega_{0,s}}(\psi) \chi_{\bar{\varphi}}^{\kappa},$$

where $\mathcal{K}_{\chi}(\cdot)$ is the Mabuchi energy on X_{can} , $\mathcal{L}_{\chi, \chi_{\bar{\varphi}}, \omega_0}(\cdot)$ is defined by

Proof The proof boils down to direct computation. First, calculate

$$\int_X \log \frac{\omega_{\bar{\varphi}}^n}{\omega^n} \omega_{\varphi}^n = t^{n-\kappa} \binom{n}{\kappa} \left(\int_X \log \frac{\chi_{\bar{\varphi}}^{\kappa}}{\chi^{\kappa}} \chi_{\bar{\varphi}}^{\kappa} + \int_{X_{can}} \left(\int_{X_y} \log \frac{\omega_{\psi}^{n-\kappa}}{\omega^{n-\kappa}} \omega_{\psi}^{n-\kappa} \right) \chi_{\bar{\varphi}}^{\kappa} + O(t) \right). \quad (6.11)$$

Also

$$\begin{aligned} & - \sum_{j=0}^{n-1} \int_X \varphi \text{Ric}(\omega) \wedge \omega^j \wedge \omega_{\varphi}^{n-j-1} \\ &= t^{n-\kappa} \sum_{j=1}^{\kappa-1} \binom{n}{\kappa} \int_{X_{can}} \bar{\varphi} (-\text{Ric}(\chi) + \omega_{WP}) \wedge \chi^j \wedge \chi_{\bar{\varphi}}^{\kappa-j-1} \\ &\quad - t^{n-\kappa} \sum_{j=0}^{n-\kappa-1} \sum_{i=j}^{\kappa+j} A_{i,j} \int_{s \in X_{can}} \left(\int_{X_s} \text{Ric}(\omega_{0,s}) \wedge \omega_{0,s}^j \wedge \omega_{\psi,s}^{n-\kappa-1-j} \right) \chi^i \wedge \chi_{\bar{\varphi}}^{\kappa-i} + O(t^{n-\kappa+1}) \end{aligned}$$

and

$$\sum_{j=0}^n \int_X \varphi \omega_t^j \wedge \omega_{\varphi}^{n-j} = \binom{n+1}{\kappa+1} t^{n-\kappa} \sum_{j=0}^{\kappa} \int_{X_{can}} \bar{\varphi} \chi^j \wedge \chi_{\bar{\varphi}}^{\kappa-j} + O(t^{n-\kappa+1}). \quad (6.12)$$

The theorem follows from straightforward calculation by combining the above formulas.

$$\begin{aligned} & \mathcal{K}_{\omega_t}(\varphi) \\ &= \binom{n}{\kappa} t^{n-\kappa} \left(\int_{X_{can}} \log \frac{\chi_{\bar{\varphi}}^{\kappa}}{\chi^{\kappa}} \chi_{\bar{\varphi}}^{\kappa} - \int_{X_{can}} \bar{\varphi} (\text{Ric}(\chi) - \omega_{WP}) \wedge \chi^j \wedge \chi_{\bar{\varphi}}^{\kappa-j} + \frac{\kappa \bar{\mu}}{\kappa+1} \int_{X_{can}} \bar{\varphi} \chi^j \wedge \chi_{\bar{\varphi}}^{\kappa-j} \right) \\ &\quad + \binom{n}{\kappa} t^{n-\kappa} \int_{s \in X_{can}} \left(\int_{X_s} \log \frac{\omega_{\psi, s}^{n-\kappa}}{\omega_{0, s}^{n-\kappa}} \omega_{\psi, s}^{n-\kappa} \right) \chi_{\bar{\varphi}}^{\kappa} \\ &\quad - t^{n-\kappa} \sum_{j=0}^{n-\kappa-1} \sum_{i=j}^{m+j} \int_{s \in X_{can}} \left(\int_{X_s} \text{Ric}(\omega_{0,s}) \wedge \omega_{0,s}^j \wedge \omega_{\psi,s}^{n-\kappa-1-j} \right) \chi^{i-j} \wedge \chi_{\bar{\varphi}}^{m-i+j} \\ &\quad + O(t^{n-\kappa+1}), \end{aligned}$$

where $\bar{\mu} = \frac{(2\pi c_1(X_{can}) - [\omega_{WP}]) \cdot [\chi]^{\kappa-1}}{[\chi]^\kappa}$.

□

We also investigate the asymptotics of the Mabuchi energy in the case of a fibred space studied by Fine in [Fi].

Let $f : X \rightarrow \Sigma$ be a Kähler surface admitting a non-singular holomorphic fibration over Σ , with fibres of genus at least 2. We also assume $c_1(\Sigma) < 0$. Let V be the vertical tangent bundle of X and $[\omega_t] = -f^*c_1(\Sigma) - tc_1(V)$.

Let $\chi, \omega_0 \in -c_1(V)$ and ω_t be defined as in Section 6.1. We consider the asymptotics of the Mabuchi energy $\mathcal{K}_{\omega_t}(\cdot)$ as t tends to 0.

Theorem 6.3 *Let $\omega_0 \in -c_1(V)$ be a closed $(1, 1)$ such that its restriction on each fibre is a hyperbolic metric. Let $\omega_t = \chi + t\omega_0$ and $\omega_\varphi = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi$ be a metric deformation, where $\varphi \in C^\infty(\Sigma)$. Then we have*

$$\mathcal{K}_{\omega_t}(\varphi) = 2t\mathcal{K}_{\chi, \theta}(\varphi) + O(t^2). \quad (6.13)$$

Theorem 6.2 and Theorem 6.3 can be considered as an adjunction type formula for the Mabuchi energy on an algebraic fibre space.

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References

- [Au] Aubin, T. *Equations du type Monge-Ampère sur les variétés Kähleriennes compacts*, Bull. Sc. Math. 102 (1976), 119-121.
- [BaMu] Bando, S. and Mabuchi, T., *Uniqueness of Einstein Kähler metrics modulo connected group actions*, Algebraic geometry, Sendai, 1985, 11-40, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.
- [BiCaHaMc] Birkar, C., Cascini, P., Hacon C. and McKernan, J., *Existence of minimal models for varieties of log general type*, arXiv:math/0610203.
- [Ca] Cao, H., *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math. 81 (1985), no. 2, 359-372.
- [Ch] Chow, B., *The Ricci flow on the 2-sphere*, J. Differential Geom. 33 (1991), no. 2, 325-334.
- [ChTi] Chen, X.X. and Tian, G., *Ricci flow on Kähler-Einstein surfaces*, Invent. Math. 147 (2002), no. 3, 487-544.
- [ChYa] Cheng, S. Y. and Yau, S. T., *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. 28 (1975), no. 3, 333-354.
- [ClKoMo] Clemens, H., Kollar, J. and Mori, S., *Higher-dimensional complex geometry*, Astérisque No. 166 (1988), 144 pp. (1989).

- [DePa] Demailly, J-P. and Pali, N., *Degenerate complex Monge-Ampère equations over compact Kähler manifolds*, arXiv:0710.5109.
- [DiZh] Dinew, S. and Zhang, Z., *Stability of Bounded Solutions for Degenerate Complex Monge-Ampère equations*, arXiv:0711.3643.
- [Do] Donaldson, S. K., *Scalar curvature and projective embeddings, I.*, J. Differential Geom. 59 (2001), no. 3, 479-522.
- [EyGuZe1] Eyssidieux, P., Guedj, V. and Zeriahi, A., *Singular Kähler-Einstein metrics*, preprint, math.AG/0603431.
- [EyGuZe2] Eyssidieux, P., Guedj, V. and Zeriahi, A., *A priori L^∞ -estimates for degenerate complex Monge-Ampère equations*, preprint, arXiv:0712.3743.
- [FaLu] Fang, H. and Lu, Z., *Generalized Hodge metrics and BCOV torsion on Calabi-Yau moduli*, J. Reine Angew. Math. 588 (2005), 49-69.
- [Fi] Fine, J., *Constant scalar curvature Kähler metrics on fibred complex surfaces*, J. Differential Geom. 68 (2004), no. 3, 397-432.
- [GrWi] Gross, M. and Wilson, P. M. H., *Large complex structure limits of K3 surfaces*, J. Differential Geom. 55 (2000), no. 3, 475-546.
- [Ha] Hamilton, R., *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982), no. 2, 255-306.
- [Kol1] Kolodziej, S., *The complex Monge-Ampère equation*, Acta Math. 180 (1998), no. 1, 69-117.
- [Kol2] Kolodziej, S., *Stability of solutions to the complex Monge-Ampère equation on compact Kähler manifolds*, preprint.
- [LiYa] Li, P. and Yau, S.T., *Estimates of eigenvalues of a compact Riemannian manifold*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), pp. 205-239, Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980.
- [La] Lazarsfeld, J., *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*, A Series of Modern Surveys in Mathematics, 48. Springer-Verlag, Berlin, 2004. xviii+387 pp.
- [Lo] Lott, J., *On the long-time behavior of type-III Ricci flow solutions*, Math. Ann. 339 (2007), no. 3, 627-666.
- [Pe] Perelman, P., *The entropy formula for the Ricci flow and its geometric applications*, preprint math.DG/0211159.
- [PhSt] Phong, D. H. and Sturm, J., *On stability and the convergence of the Kähler-Ricci flow*, J. Differential Geom. 72 (2006), no. 1, 149-168.
- [Si1] Siu, Y-T., *Multiplier ideal sheaves in complex and algebraic geometry*, Sci. China Ser. A 48 (2005), suppl., 1-31.
- [Si2] Siu, Y-T., *A General Non-Vanishing Theorem and an Analytic Proof of the Finite Generation of the Canonical Ring*, arXiv:math/0610740.
- [SoTi] Song, J. and Tian, G., *The Kähler-Ricci flow on minimal surfaces of positive Kodaira dimension*, Invent. Math. 170 (2007), no. 3, 609-653.
- [SoWe] Song, J. and Weinkove, B., *On the convergence and singularities of the J-flow with applications to the Mabuchi energy*, Comm. Pure Appl. Math. 61 (2008), 210-229.
- [StYaZa] Strominger, A., Yau, S.T. and Zaslow, E., *Mirror symmetry is T-duality*, Nuclear Phys. B 479 (1996), no. 1-2, 243-259.
- [Ti1] Tian, G., *On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$* , Invent. Math. 89 (1987), no. 2, 225-246.
- [Ti2] Tian, G., *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math. 101, no. 1 (1990), 101-172.

- [Ti3] Tian, G., *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom. 32 (1990), no. 1, 99-130.
- [Ti4] Tian, G., *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric*, Mathematical aspects of string theory (San Diego, Calif., 1986), World Sci. Publishing, Singapore, 629-646.
- [TiZha] Tian, G. and Zhang, Z., *A note on the Kähler-Ricci flow on projective manifolds of general type*, preprint.
- [TiZhu] Tian, G. and Zhu, X., *Convergence of Kähler Ricci flow*, J. Amer. Math. Soc. 20 (2007), no. 3, 675-699.
- [To] Tosatti, V., *Limits of Calabi-Yau metrics when the Kahler class degenerates*, arXiv:0710.4579.
- [Ts1] Tsuji, H., *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann. 281 (1988), 123-133.
- [Ts2] Tsuji, H., *Analytic Zariski decomposition*, Proc. of Japan Acad. 61 (1992), 161-163.
- [Ts3] Tsuji, H., *Generalized Bergmann Metrics and Invariance of Plurigenera*, arXiv:math/960448.
- [Ue] Ueno, K., *Classification theory of algebraic varieties and compact complex spaces*, notes written in collaboration with P. Cherenack, Lecture Notes in Mathematics, Vol. 439, Springer-Verlag, Berlin-New York, 1975. xix+278 pp.
- [Ya1] Yau, S.T., *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math. 100 (1978), no. 1, 197-203.
- [Ya2] Yau, S.T. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. 31 (1978), 339-411.
- [Zh] Zhang, Z., *On degenerate Monge-Ampère equations over closed Kähler manifolds*, Int. Math. Res. Not. 2006, Art. ID 63640, 18 pp.

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